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THE QUARTERLY JOURNAL OF
MATHEMATICS
(OXFORD SERIES)

Volume 10 *No.* 40 *December* 1939

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OXFORD
AT THE CLARENDON PRESS
1939

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS
Entered as second class at the New York, U.S.A., Post Office

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, J. HODGKINSON, E. G. C. POOLE
With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY, A. E. H. LOVE, E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published in March, June, September, and December, at a price of 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E.C. 4).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C.4

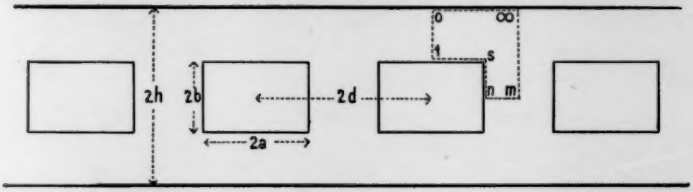
NOTE ON A PROBLEM IN ELECTROSTATICS

By E. P. ADAMS (*Princeton*)

[Received 11 February 1939]

1. THE ingenious method used by Richmond (1) for getting the solution of problems in electrostatics connected with a grating of rounded bars may be applied to the problem of such a grating placed midway between two infinite planes at the same potential.

We begin with a grating of cylinders of rectangular cross-section as shown in the figure which represents the z-plane transverse to the



axes of the cylinders. The distance between the two parallel planes is $2h$, and $2d$ is the distance between the axes of consecutive cylinders, each of dimensions $2a$ along the x-axis and $2b$ along the y-axis. On account of symmetry we need consider only the area within the dotted lines. This area is transformed to the upper half of a t-plane by the Schwarz-Christoffel method, the real values of t at the corners being $0 < 1 < s < n < m < \infty$. The differential equation is

$$\frac{dz}{dt} = \frac{A(t-s)^{\frac{1}{2}}}{\{t(t-1)(t-n)(t-m)\}^{\frac{1}{2}}}.$$

Following the method of Richmond, we put

$$A(t-s)^{\frac{1}{2}} = P(t-n)^{\frac{1}{2}} + Q(t-1)^{\frac{1}{2}},$$

and get

$$z = P\int \frac{dt}{\{t(t-1)(t-m)\}^{\frac{1}{2}}} + Q\int \frac{dt}{\{t(t-n)(t-m)\}^{\frac{1}{2}}} + B,$$

where B is a constant. In the first integral put

$$t = \frac{1}{\mathrm{dn}^2 w}$$

with a modulus k determined by

$$k^2 = 1 - \frac{1}{m},$$

and with corresponding values of t and w given by

$$\begin{array}{lccccc} t = & 0 & 1 & m & \infty \\ w = & iK' & 0 & K & K+iK'. \end{array}$$

For $t = n$, since $1 < n < m$, $w = u_0$ where u_0 is real and lies between 0 and K, and is determined by

$$\mathrm{dn}^2 u_0 = \frac{1}{n}.$$

In the second integral put

$$t = \frac{n}{\mathrm{dn}^2 w_1}$$

with a modulus k_1 determined by

$$k_1^2 = 1 - \frac{n}{m}.$$

Corresponding values of t and w_1 are given by

$$\begin{array}{lccccc} t = & 0 & n & m & \infty \\ w_1 = & iK_1' & 0 & K_1 & K_1+iK_1'. \end{array}$$

For $t = 1$, $w = iv_0$ where v_0 is real, lies between 0 and K_1', and is given by

$$\mathrm{dn}^2(v_0, k_1') = \frac{k_1^2}{k^2}.$$

We can now write

$$z = P'w + Q'w_1 + B,$$

and the constants are determined from the table,

$$\begin{array}{lccccc} t = & 0 & 1 & n & m & \infty \\ z = & ih & ib & a & d & d+ih \\ w = & iK' & 0 & u_0 & K & K+iK' \\ w_1 = & iK_1' & iv_0 & 0 & K_1 & K_1+iK_1' \end{array}$$

On substituting these values we get

$$B = 0, \qquad d = P'K + Q'K_1, \qquad b = Q'v_0,$$
$$a = P'u_0, \qquad h = P'K' + Q'K_1'.$$

If we take $a = b$ so as to have an approximation to circles, we have

$$\frac{Q'}{P'} = \frac{u_0}{v_0},$$
$$\frac{d}{a} = \frac{1}{u_0}\left(K + \frac{u_0}{v_0}K_1\right),$$
$$\frac{h}{a} = \frac{1}{u_0}\left(K' + \frac{u_0}{v_0}K_1'\right),$$

and u_0, v_0 are determined by

$$\mathrm{dn}^2(u_0, k) = \frac{k'^2}{k_1'^2}, \qquad \mathrm{sn}^2(u_0, k) = \frac{k^2-k_1^2}{k^2k_1'^2},$$

$$\mathrm{dn}^2(v_0, k_1') = \frac{k_1^2}{k^2}, \qquad \mathrm{sn}^2(v_0, k_1') = \frac{k^2-k_1^2}{k^2k_1'^2}.$$

By assuming values for k and k_1 ($< k$) the ratios d/a and h/a may thus be found.

On the cylinder, $1 \leqslant t \leqslant n$, so that $w = u$ with $0 \leqslant u \leqslant K$, and $w_1 = iv$ with $0 \leqslant v \leqslant v_0$. Put

$$t = 1+(n-1)\lambda,$$

so that $\lambda = 0$, 1 for $t = 1$, n. Then the equation of the positive quadrant of the cross-section of the cylinder may be written

$$\frac{x}{a} = \frac{u}{u_0}, \qquad \frac{y}{a} = \frac{v}{v_0},$$

and u, v are given by

$$\mathrm{dn}^2(u, k) = \frac{k'^2}{k'^2-(k^2-k_1^2)\lambda},$$

$$\mathrm{dn}^2(v, k_1') = \frac{k_1^2}{k^2-(k^2-k_1^2)\lambda}.$$

For a circle we should have

$$\frac{x^2+y^2}{a^2} = \frac{r^2}{a^2} = 1.$$

Milne-Thomson's tables (**2**) are useful for rapidly getting a numerical solution. As an example, take

$$k^2 = k'^2 = 0{\cdot}5; \qquad k_1^2 = 0{\cdot}1, \quad k_1'^2 = 0{\cdot}9.$$

Then we find $u_0 = 1{\cdot}3821$, $v_0 = 1{\cdot}6327$,

and $d/a = 2{\cdot}3291$, $h/a = 2{\cdot}9206$.

For the shape of the cylinders we find

$\lambda = 0$	0·1	0·3	0·5	0·7	0·9	1·0
$u = 0$	0·4003	0·6969	0·9098	1·0968	1·2819	1·3821
$v = 1{\cdot}6327$	1·5677	1·4239	1·2505	1·0196	0·6232	0
$x/a = 0$	0·2896	0·5043	0·6583	0·7937	0·9276	1
$y/a = 1$	0·9603	0·8722	0·7659	0·6245	0·3818	0
$r/a = 1$	1·0030	1·0072	1·0097	1·0095	1·0031	1

Thus for all practical purposes these cylinders can be regarded as circular.

If $k = k_1'$ and $k' = k_1$, with $k_1 < k$, we get $h/a = d/a$, $u_0 = v_0$, $u = v$. The cylinders are now symmetrical not only with respect to the x- and y-axes, but also with respect to axes inclined to these at an angle of 45°. Thus with $k^2 = 0{\cdot}9$ we find

$$h/a = d/a = 1{\cdot}8773,$$

and the greatest value of r/a, for $\lambda = 0{\cdot}5$, is $r/a = 1{\cdot}034$. For $k^2 = 0{\cdot}8$ we find

$$h/a = d/a = 2{\cdot}2595,$$

and the greatest value of r/a is $1{\cdot}015$.

For k and k_1 small, less than about 10^{-3}, we can put

$$K = K_1 = \tfrac{1}{2}\pi,$$

$$K' = \log\frac{4}{k}, \qquad K_1' = \log\frac{4}{k_1},$$

$$\mathrm{sn}^2 u_0 = \sin^2 u_0 = 1 - \frac{k_1^2}{k^2},$$

$$\mathrm{sn}^2(v_0, k_1') = \tanh^2 v_0 = \sin^2 u_0,$$

so that

$$\sec u_0 = \cosh v_0,$$

$$\frac{d}{a} = \frac{\pi(u_0 + v_0)}{2u_0 v_0},$$

$$\frac{h}{a} = \frac{1}{u_0}\left(\log\frac{4}{k} + \frac{u_0}{v_0}\log\frac{4}{k_1}\right).$$

For $k = k_1 = 0$, $h/a = \infty$ and we have the case considered by Richmond of a grating of rounded bars in free space.

2. Let Q be the charge on unit length of each cylinder at potential V, the two infinite planes being at potential 0. The complex potential is

$$\chi = \phi + i\psi,$$

with ϕ as the real potential. The line of force from $t = 0$ to $t = 1$ is taken as $\psi = 0$. Then the line of force from $t = n$ to $t = \infty$ is $\psi = \pi Q$. The equipotential from $t = 1$ to $t = n$ is $\phi = V$, and from $t = \infty$ to $t = 0$ it is $\phi = 0$. The χ-plane is now the interior of a rectangle, and to transform to the upper half of the t-plane we have

$$\frac{d\chi}{dt} = \frac{C}{\{t(t-1)(t-n)\}^{\frac{1}{2}}}.$$

Put $$t = \frac{1}{\mathrm{dn}^2 w_2},$$

with a modulus k_2 determined by

$$k_2^2 = \frac{n-1}{n} = \frac{k^2-k_1^2}{k_1'^2}.$$

We then get $$\chi = C'w_2 + D,$$

and the constants are determined by

$t =$	0	1	n	∞
$w_2 =$	iK_2'	0	K_2	K_2+iK_2'
$\chi =$	0	V	$V+i\pi Q$	$i\pi Q$.

We then get $$\chi = \frac{i\pi Q}{K_2} w_2 + V,$$

and $$\frac{Q}{V} = \frac{K_2}{\pi K_2'}.$$

The last equation gives the ratio of the charge on unit length of each cylinder to the difference of potential between the cylinders and the planes. With

$$q_2 = e^{-\frac{\pi K}{K_2}}, \qquad q_2' = e^{-\frac{\pi K_2'}{K_2}},$$

we get $$\frac{Q}{V} = \frac{1}{\log 1/q_2} = \frac{1}{\pi^2} \log \frac{1}{q_2'}.$$

In the example considered, with $k^2 = 0{\cdot}5$, $k_1^2 = 0{\cdot}1$, we get

$$k_2^2 = 4/9,$$

and $$\frac{Q}{V} = 0{\cdot}3025.$$

By giving different values to the ratio b/a the exact solution may thus be obtained for a grating of cylinders of a variety of shapes placed midway between two parallel planes at the same potential.

3. If we solve the problem of an infinite series of line charges Q at a distance $2d$ apart, placed midway between two infinite planes at the same potential at a distance $2h$ apart, and assume that cylinders of radius a with the line charges as axes are the equipotential surfaces V, we shall find

$$\frac{Q}{V} = 1 \Big/ \left(2 \log \frac{d}{k^{\frac{1}{4}} K a}\right),$$

with a modulus determined by

$$\log\frac{1}{q}=\frac{2\pi h}{d}.$$

For the example considered, with $h/a = 2{\cdot}9206$, $d/a = 2{\cdot}3291$, we find

$$\frac{Q}{V}=0{\cdot}2995,$$

which is only about 1 per cent. less than the value we found for the approximately circular cylinders (0·3025).

4. The numerical results obtained by Richmond may be used to solve the problem of a single cylinder midway between two infinite planes at the same potential. This is the condenser problem that is solved for a circular cylinder by Love (**3**) in a recent paper in this *Journal*. It is readily found that the capacity S is given by

$$S=\frac{K}{\pi K'},$$

where the modulus is $k=\sin\beta,$

and $$\cosh b=\sec\beta, \qquad \frac{a}{h}=\frac{2b\beta}{\pi(b+\beta)}.$$

With the values of β given by Richmond (except for the last for which I find $b = 3{\cdot}9045$, $\beta = 87^\circ\ 41'\ 25''$) we get the following table:

$a/h =$	0·1	0·2	0·5	0·7
$S =$	0·1965	0·2702	0·5433	0·9318
$C =$	0·1965	0·2701	0·5381	0·8684
$S' =$	0·1965	0·2701	0·5349	0·8339

The third line gives the values of the capacity C, which were found by Love for accurately circular cylinders. The last line gives the values of the capacity if we assume that the circular cylinder of radius a with a line charge Q midway between the planes as an axis is the equipotential V. The capacity in this case may be shown to be

$$S'=1\Big/\left(2\log\frac{4h}{\pi a}\right).$$

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2. L. M. Milne-Thomson, *Die elliptischen Funktionen von Jacobi* (Berlin, 1931).
3. A. E. H. Love, *Quart. J. of Math.* (Oxford), 9 (1938), 246.

TRANSFORMATIONS OF ENUMERABLE SETS WHICH ARE DENSE IN AN INTERVAL

R. COOPER (*Belfast*)

[Received 28th March 1939]

1. ONE-ONE bicontinuous transformations of enumerable sets which are dense in given intervals have been studied by several writers. Among their results are the following:

(i)* Let E, F be enumerable sets, each dense in Euclidean space of n dimensions. Then there is a one-one transformation $P = f(p)$ in which P, p are points of E, F respectively such that f and its inverse both satisfy a Lipschitz condition, i.e. there is a constant K such that

$$\frac{1}{K} < \frac{\rho(P_1, P_2)}{\rho(p_1, p_2)} < K,$$

where (P_1, p_2), (P_2, p_2) are pairs of corresponding points and $\rho(a, b)$ denotes the distance between the points a, b.

(ii) W. Sierpinski† says that a set F is smaller-(N), or smaller in von Neumann's sense, than a set E if there is a function $f(x)$ defined in E which sets up a one-one correspondence between E, F and satisfies the condition that $|f(x)-f(y)| < |x-y|$ for each pair of distinct points x, y of E. He proved that, if E, F are linear enumerable sets contained in and dense in intervals of lengths a, b and if $b < a$, the set F is smaller-(N) than E. It is obvious that F cannot be smaller-(N) than E if $b > a$.

2. In this paper we are concerned primarily with the case not covered by Sierpinski's result, i.e. when the intervals are of equal length. Our result when combined with a linear transformation throws light on the transformations which can be constructed in the case of unequal intervals. If either of the intervals is unbounded, separate consideration is needed, but it will be sufficient to mention briefly the differences from the argument of the primary case. In all cases it appears that a very smooth relation can be established between any pair of enumerable sets dense in intervals. The following theorem gives the principal result.

* P. Urysohn, 'Sur les multiplicités Cantoriennes', *Fund. Math.* 7 (1925), 29–137, esp. 83–9.

† W. Sierpinski, 'Sur une propriété des ensembles dénombrables denses dans un intervalle', *Mathematica*, 11 (1935), 222–8.

THEOREM. *Let E, F be enumerable sets contained in and dense in the interval $(0, 1)$. Let ϵ (< 1) be a small positive number and let N be any integer. Then there exists a function $f(x)$ defined for $0 \leqslant x \leqslant 1$ which sets up a one-one correspondence between E, F and satisfies the following conditions:*

(i) *$f(x)$ is convex and increasing in $(0, 1)$;*
(ii) *$f^{(\nu)}(x)$ exists* for $0 \leqslant x \leqslant 1$ and $\nu \leqslant N$;*
(iii) *$1-\epsilon < f'(x) < 1+\epsilon$ for $0 \leqslant x \leqslant 1$;*
(iv) *$|f^{(\nu)}(x)| < \epsilon$ for $\nu = 2, 3, \ldots, N$ and $0 \leqslant x \leqslant 1$.*

We construct this function as the limit of a decreasing sequence of functions $f_n(x)$ each of which sets up a one-one correspondence between n pairs of points of E, F. If N is unity,† it is possible to take these approximating functions as polygonal functions,‡ but, if on such lines we try to satisfy conditions about the second derivative, great difficulties arise, partly on account of the discontinuities of the derivative of the approximating functions. We therefore use as approximations functions which have no discontinuities among their derivatives up to the Nth order.

The construction arranges the sets E, F as sequences $\{x_n\}$, $\{y_n\}$, and we are able to define a function $f_n(x)$ to satisfy the following conditions in $(0, 1)$:

(i) $f_n(x)$ is convex and increasing;
(ii) $f_n^{(\nu)}(x)$ exists and is continuous for $\nu \leqslant N$;
(iii) $1-\epsilon+2^{-n-1}\epsilon < f_n'(x) < 1+\epsilon-2^{-n-1}\epsilon$;
(iv) $|f_n^{(\nu)}(x)| < (1-2^{-n-1})\epsilon$ for $2 \leqslant \nu \leqslant n$;
(v) the lower bound of $f_n''(x)$ is not zero;
(vi) $f_n(0) = 0$, $f_n(1) = 1$, $f_n(x_r) = y_r$ $(r = 1, 2, \ldots, n)$.

We also are able to arrange that, for $0 \leqslant x \leqslant 1$,

$$|f_n(x)-f_{n+1}(x)| < 2^{-n-2}\epsilon,$$

and

$$|f_n^{(\nu)}(x)-f_{n+1}^{\nu}(x)| < 2^{-n-2}\epsilon \quad (1 \leqslant \nu \leqslant N).$$

From these it follows that the sequences $\{f_n(x)\}$, $\{f_n^{(\nu)}(x)\}$ $(\nu \leqslant N)$ are uniformly convergent in $0 \leqslant x \leqslant 1$ and $f(x) = \lim f_n(x)$ has the properties stated in the theorem.

* The derivatives at 0 and 1 are right- and left-hand derivatives.

† This was the case I originally considered, and I am indebted to Dr. J. Gillis for suggesting that it might be possible to impose condition (iv).

‡ This is on the lines of Urysohn, loc. cit.

3. We now give the details of the construction. The curve $y = f_0(x)$ is not required to pass through any points (x_r, y_r). We therefore choose it to be a circular arc.

To construct the function $f_{n+1}(x)$ from $f_n(x)$ we require the following lemma:

LEMMA. *Given any positive number* δ, *a function* $\phi(x)$ *can be defined which will satisfy the following conditions in an interval* (α, β):

(i) $\phi(\alpha) = 0, \quad 0 < \phi(\beta) < \delta$;
(ii) $\phi^{(\nu)}(\alpha) = \phi^{(\nu)}(\beta) = 0$ *for* $1 \leqslant \nu \leqslant N$;
(iii) $0 < \phi'(x) < \delta$ *for* $\alpha < x < \beta$;
(iv) $|\phi^{(\nu)}(x)| < \delta$ *for* $2 \leqslant \nu \leqslant N$, $\alpha < x < \beta$;
(v) $\phi^{(N)}(x)$ *is continuous.*

For this it is enough to observe that, if k is sufficiently small,

$$\phi(x) = k \int_{\alpha}^{x} (t-\alpha)^N(\beta-t)^N \, dt$$

satisfies the conditions.

Since the sets E, F are enumerable, we can arrange them as sequences $\{\xi_\nu\}$, $\{\eta_\nu\}$ say, The curve $y = f_n(x)$ passes through the points (x_r, y_r) $(r = 1, 2, \ldots, n)$ whose coordinates are certain terms of the sequences $\{\xi_r\}$, $\{\eta_r\}$.

When n is odd, we take x_{n+1} to be the first term of $\{\xi_\nu\}$ which is not one of the numbers $x_1, x_2, \ldots, x_n$. The numbers $x_1, \ldots, x_n$ divide $(0, 1)$ into $n+1$ intervals; let (x', x'') be the one which contains x_{n+1}, and let $y' = f(x')$, $y'' = f(x'')$. We choose for y_{n+1} the first term in $\{\eta_\nu\}$ which lies between y' and y'' and is such that there is a function $f_{n+1}(x)$ with the following properties:

(i) $f_{n+1}(x) = f_n(x)$ except for $x' < x < x''$, and
$f_{n+1}(x) < f_n(x)$ for $x' < x < x''$;
(ii) $f_{n+1}(x_{n+1}) = y_{n+1}$;
(iii) $f^{(N)}_{n+1}(x)$ exists and is continuous for $0 \leqslant x \leqslant 1$;
(iv) $f_{n+1}(x)$ is convex and strictly increasing;
(v) $f^{(\nu)}_{n+1}(x) = f^{(\nu)}_n(x)$ at x', x_{n+1}, and x'' $(1 \leqslant \nu \leqslant N)$;
(vi) $|f^{(\nu)}_{n+1}(x) - f^{(\nu)}_n(x)| < \delta_n \quad (0 \leqslant \nu \leqslant N)$ where δ_n is a prescribed number which is less than $2^{-n-2}\epsilon$. The lemma shows that this is possible provided that δ_n is less than the lower bounds of $f'_n(x)$ and $f''_n(x)$.

When n is even, we choose y_{n+1} as the first term of $\{\eta_\nu\}$ which is

not one of $y_1, \ldots, y_n$, and then x_{n+1} to be the first available term of $\{\xi_\nu\}$ so that the same conditions are satisfied as in the case of n odd. This completes the construction; it is obvious that the sequences $\{x_n\}$, $\{y_n\}$ contain every term of the sets E, F.

4. We see at once that, if the given sets E, F are contained in and dense in different intervals, there is a function transforming E into F which has the same properties as the function of the theorem save that instead of the condition (iii) its derivative lies in a prescribed neighbourhood of the ratio of the lengths of the intervals.

5. Unbounded Sets. If both the sets E, F are dense in $(-\infty, \infty)$, we can construct a function with the properties of the one in the theorem save that they hold for all values of x instead of in $(0, 1)$. Only slight modifications of the above argument are needed; if k is sufficiently small, we can take for a first approximating function

$$f_0(x) = \tfrac{1}{2}\{x + \sqrt{(1+k^2x^2)}\},$$

whose graph is one branch of a hyperbola with asymptotes having gradients $\frac{1}{2}(1 \pm k)$.

We could similarly require the derivative to differ by not more than ϵ from any selected positive number instead of from unity.

If the intervals containing E, F are (a, ∞), (b, ∞), we can satisfy the same conditions as in the case of the intervals $(-\infty, \infty)$, $(-\infty, \infty)$.

6. Further differences appear in other cases. First suppose that the intervals containing E, F are $(-\infty, \infty)$ and (a, b), where a, b are finite. Then we can select arbitrarily a point c and a small number ϵ, and show that there exists a function $f(x)$ such that $y = f(x)$ sets up a one-one correspondence between x in E and y in F and such that

(i) $f(x)$ is strictly increasing;
(ii) $f(x)$ is convex in $(-\infty, c)$ and concave in (c, ∞);
(iii) $f^{(\nu)}(x)$ exists and is continuous for all x for $\nu \leqslant N$;
(iv) $|f^{(\nu)}(x)| < \epsilon \quad (1 \leqslant \nu \leqslant N)$.

The argument is similar to those above. We can construct a first approximation satisfying the conditions (i)–(iv) with $\frac{1}{2}\epsilon$ in place of ϵ, e.g. if k is sufficiently small, we can take

$$f_0(x) = \frac{b+a}{2} + \frac{b-a}{\pi}\tan^{-1}k(x-c).$$

If c is not in E but $\frac{1}{2}(b+a)$ is in F, we take $y_1 = \frac{1}{2}(b+a)$. If c is in E but $\frac{1}{2}(b+a)$ is not in F, we take $x_1 = c$ and reverse the rules above

about odd and even suffixes. If c is in E and $\frac{1}{2}(b+a)$ is in F, we take $x_1 = c$ and $y_1 = \frac{1}{2}(b+a)$ and $f_1(x) = f_0(x)$.

In addition to using the lemma as above we have to apply it as follows. If x_{n+1} is to the left of all the points $x_1, x_2, \ldots, x_n$, we take a number $X < x_{n+1}$ and apply the lemma to the interval (X, x_{n+1}), because f_{n+1} and f_n must tend to the same limits as $x \to +\infty$ and $-\infty$. The case of x_{n+1} to the right of all of $x_1, \ldots, x_n$ is treated similarly.*

7. If F is in an interval (a, ∞) and E in $(-\infty, \infty)$, we can find a function which transforms E to F, satisfies the conditions (i), (iii), (iv) for $f(x)$ in § 6, and is convex in $(-\infty, \infty)$. To see this it is sufficient to note that a suitable first approximation is

$$a + kx + \sqrt{(k^2x^2+1)}$$

if k is small enough.

8. If E is in (a, ∞) and F in (b, c), there is a function $f(x)$ transforming E into F which satisfies

(i) $f(x)$ is concave and increasing for $a < x$;
(ii) $f^{(\nu)}(x)$ exists for $a \leqslant x$ and $\nu \leqslant N$;
(iii) $|f^{(\nu)}(x)| < \epsilon$ for $a \leqslant x$.

For this we need only note that

$$b + \frac{2(c-b)}{\pi} \tan^{-1} k(x-a)$$

is a suitable first approximation if k is small enough.

9. The only cases not examined above are those in which one of the intervals is of the type $(-\infty, a)$, but a reflection reduces these to cases already dealt with.

It is possible to establish similar results for sets in space of any number of dimensions, but the results and methods are so similar to those used above that it is unnecessary to state them.

Our results show that the possible degree of smoothness of the transformation function can give no method of assigning any relation of relative density to a pair of enumerable sets.†

* This step was not needed in § 5, because there, in the case of

$$x_{n+1} < x_i \quad (i = 1, \ldots, n),$$

we could take $f_{n+1} - f_n$ constant in $(-\infty, x_{n-1})$.

† I am indebted to the referee for some suggestions which have enabled me to make some points clearer.

ON SELF-RECIPROCAL FUNCTIONS

By B. MOHAN (*Benares*)

[Received 1 May 1939]

1. A FEW years back I proved some theorems concerning self-reciprocal functions. The object of this note is to give some more general theorems of the same type and to derive some new self-reciprocal functions therefrom.

I shall say that a function is R_ν or $-R_\nu$ according as it is self-reciprocal or skew-reciprocal for J_ν transforms.

Hardy and Titchmarsh have defined a class of functions $A(\alpha, a)$, where $0 < \alpha \leqslant \pi$, $a < \frac{1}{2}$, as functions (i) which are analytic functions of $x = re^{i\theta}$ regular in the angle A defined by $r > 0$, $|\theta| < \alpha$, and (ii) which are $O(|x|^{-a-\delta})$ for small x, and $O(|x|^{a-1+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqslant \alpha - \eta < \alpha$.

I take the following theorem from Hardy and Titchmarsh (**7**) concerning this class of functions:

A necessary and sufficient condition that a function $f(x)$ of $A(\alpha, a)$ should be R_ν is that it should be of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)x^{-s}\,ds, \tag{1.1}$$

where $\psi(s)$ is regular, and satisfies the equation

$$\psi(s) = \psi(1-s) \tag{1.2}$$

in the strip

$$a < \sigma < 1-a; \tag{1.3}$$

$$\psi(s) = O\{e^{(\frac{1}{4}\pi-\alpha+\eta)|t|}\} \tag{1.4}$$

for every positive η and uniformly in any strip interior to (1.3); *and c is any value of σ in* (1.3).

2. THEOREM I. *If*

$$P(x) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s)x^{-s}\,ds, \tag{2.1}$$

where $\omega(s)$ satisfies (1.2) *in the strip* (1.3), *and*

$$\omega(s) = O\{e^{(\frac{1}{4}\pi-2\alpha+2\eta)|t|}\}; \tag{2.2}$$

and

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\chi(s)x^{-s}\,ds, \tag{2.3}$$

where $\chi(s)$ *satisfies* (1.2) *in the strip* (1.3), *and*

$$\chi(s) = O\{e^{(\frac{1}{2}\pi-\alpha+\eta)|t|}\}, \tag{2.4}$$

then the function $g(x) = \int_0^\infty P(xy)f(y)\,dy$

is R_λ *and belongs to* $A(3\alpha, a)$.

By (2.1) we have

$$g(x) = \frac{1}{2\pi i}\int_0^\infty f(y)\,dy \int_{b-i\infty}^{b+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s)(xy)^{-s}\,ds \tag{2.5}$$

$$= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s)x^{-s}\,ds \int_0^\infty y^{-s}f(y)\,dy. \tag{2.6}$$

To justify the inversion of the order of integration we observe that the modulus of the repeated integral is not greater than a constant multiple of

$$x^{-b}\int_{-\infty}^{\infty} 2^b|\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}b+\tfrac{1}{2}it)||\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}b+\tfrac{1}{2}it)||\omega(s)|\,dt \times \times \int_0^\infty y^{-b}|f(y)|\,dy. \tag{2.7}$$

The inversion of the order of integration will be justified if the two integrals separately exist.

Now, by virtue of (2.2) and

$$|\Gamma(B+\tfrac{1}{2}it)| \sim Ce^{-\frac{1}{4}\pi|t|}|t|^{B-\frac{1}{2}}, \tag{2.8}$$

the integrand in the t-integral is substantially the same as

$$e^{-\frac{1}{4}\pi|t|}|t|^{-\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}b}e^{-\frac{1}{4}\pi|t|}|t|^{-\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}b}e^{(\frac{1}{2}\pi-2\alpha+2\eta)|t|},$$

i.e. the same as $e^{-2(\alpha-\eta)|t|}|t|^{-\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu+b}$.

Hence the t-integral exists for $\alpha-\eta > 0$.

Also, by virtue of (2.4) and (2.8), the modulus of the integrand in (2.3) is not greater than a constant multiple of

$$e^{-\frac{1}{4}\pi|t|}|t|^{-\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}k}e^{-\frac{1}{4}\pi|t|}|t|^{-\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}k}e^{-\frac{1}{4}\pi|t|}|t|^{\frac{1}{4}+\frac{1}{2}\lambda-\frac{1}{2}k}e^{(\frac{1}{2}\pi-\alpha+\eta)|t|}x^{-k},$$

which is the same as

$$e^{-(\alpha-\eta)|t|}|t|^{-\frac{1}{4}+\frac{1}{4}\mu+\frac{1}{4}\nu+\frac{1}{4}\lambda+\frac{1}{4}k}x^{-k}.$$

Therefore, the integral in (2.3) is uniformly convergent for $\alpha > \eta$. Also, $f(x) \sim |x|^{-k}$, and since k may be taken as near b or $1-b$ as we please, it follows that the y-integral in (2.7) also converges. Hence the inversion of the order of integration in (2.5) is justifiable.

Now, by Mellin's inversion formula (**6**), we have, from (2.3),

$$\int_0^\infty x^{s-1}f(x)\,dx = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\chi(s).$$

Thus

$$\int_0^\infty x^{-s}f(x)\,dx = 2^{\frac{1}{2}-\frac{1}{2}s}\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s),$$

since $\chi(s)$ satisfies (1.2). Therefore, from (2.6), we have

$$\begin{aligned} g(x) &= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^{s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s)x^{-s}\times \\ &\qquad \times 2^{\frac{1}{2}-\frac{1}{2}s}\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)\,ds \\ &= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\psi_1(s)x^{-s}\,ds, \end{aligned}$$

where

$$\begin{aligned} \psi_1(s) = \sqrt{2}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\times \\ \times\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\omega(s)\chi(s). \end{aligned}$$

Now $\psi_1(s)$ satisfies (1.2), and, by (2.2), (2.4), (2.8),

$$\begin{aligned} \psi_1(s) &= O\{e^{-\pi|t|+(\frac{1}{2}\pi-2\alpha+2\eta)|t|+(\frac{3}{2}\pi-\alpha+\eta)|t|}\} \\ &= O\{e^{\frac{1}{2}\pi-3\alpha+3\eta)|t|}\}. \end{aligned}$$

It follows from (1.1) that $g(x)$ belongs to $A(3\alpha, a)$, and is R_λ.

3. Example I

I start with the particular case of a formula of Goldstein (**5**):

$$\int_0^\infty x^{l-1}e^{-\frac{1}{2}x}W_{k,m}(x)\,dx = \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)},$$

where $l \pm m + \frac{1}{2} > 0$. This may be thrown into the form

$$\int_0^\infty x^{2l-1}e^{-\frac{1}{2}x^2}W_{k,m}(\tfrac{1}{2}x^2)\,dx = \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{2^{1-l}\Gamma(l-k+1)},$$

where $l \pm m + \frac{1}{2} > 0$. Putting $l = \frac{1}{2}s + \frac{1}{4}\mu + \frac{1}{4}\nu - \frac{1}{4}$, $m = \frac{1}{4}(\mu - \nu)$, $k = \frac{1}{2} + \frac{1}{4}\mu + \frac{1}{4}\nu - \frac{1}{2}\lambda$, we get

$$\int_0^\infty x^{s-1} x^{\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{4}} e^{-\frac{1}{4}x^2} W_{\frac{1}{4}+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{2}\lambda, \frac{1}{4}(\mu-\nu)}(\tfrac{1}{2}x^2)\, dx$$

$$= 2^{\frac{1}{4}s+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{4}} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}\lambda+\frac{1}{2}s)},$$

where $\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s > 0$, $\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s > 0$. Hence, by Mellin's inversion formula (6) and the theory of analytic continuation,

$$x^{\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{4}} e^{-\frac{1}{4}x^2} W_{\frac{1}{4}+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{2}\lambda, \frac{1}{4}(\mu-\nu)}(\tfrac{1}{2}x^2)$$

$$= \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} 2^{\frac{1}{4}s+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{4}} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}\lambda+\frac{1}{2}s)} x^{-s}\, ds, \quad (3.1)$$

where $p > \mu+\nu-\lambda+\frac{3}{2}$. This integral is of the same form as (2.3) with

$$\chi(s) = \frac{2^{\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{4}}}{\Gamma(\frac{1}{4}+\frac{1}{2}\lambda+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\lambda-\frac{1}{2}s)},$$

and $0 < \mu+\nu-\lambda+\frac{3}{2} < 1$. Hence, the function on the left-hand side of (3.1) may be taken as the function $f(x)$ in the above theorem.

In 1932 I gave three examples of functions of the type of $P(x)$ in the above theorem (9):

(a) $x^{\frac{1}{2}(\mu+\nu+1)} K_{\frac{1}{2}(\nu-\mu)}(x)$,

where $K_m(x)$ denotes Bessel's function of the second kind of imaginary argument and order m;

(b) $x^{\frac{1}{2}(\nu-\mu+1)} J_{\frac{1}{2}(\mu+\nu)}(x)$;

(c) $x^{\frac{1}{2}(\mu-\nu+1)} J_{\frac{1}{2}(\mu+\nu)}(x)$.

I now proceed to derive self-reciprocal functions. Taking the function (b) given above as our $P(x)$, we have

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}(\nu-\mu+1)} J_{\frac{1}{2}(\mu+\nu)}(xy) y^{\frac{1}{2}(\mu+\nu-1)} e^{-\frac{1}{4}y^2} W_{\frac{1}{4}+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{2}\lambda, \frac{1}{4}(\mu-\nu)}(\tfrac{1}{2}y^2)\, dy$$

$$= x^{\frac{1}{2}(\nu-\mu+1)} \int_0^\infty y^\nu e^{-\frac{1}{4}y^2} J_{\frac{1}{2}(\mu+\nu)}(xy) W_{\frac{1}{4}+\frac{1}{4}\mu+\frac{1}{4}\nu-\frac{1}{2}\lambda, \frac{1}{4}(\mu-\nu)}(\tfrac{1}{2}y^2)\, dy.$$

Evaluating this integral by a formula given by Varma (12), we have

$$g(x) = 2^{\frac{1}{4}\nu-\frac{1}{4}\mu-\frac{1}{4}} x^{\nu+\frac{1}{2}} {}_2F_2\left(\begin{matrix} 1+\frac{1}{2}\mu+\frac{1}{2}\nu, 1+\nu \\ 1+\frac{1}{2}\mu+\frac{1}{2}\nu, 1+\frac{1}{2}\nu+\frac{1}{2}\lambda \end{matrix}; -\tfrac{1}{2}x^2\right)$$

$$= 2^{\frac{1}{4}\nu-\frac{1}{4}\mu-\frac{1}{4}} x^{\nu+\frac{1}{2}} {}_1F_1(1+\nu, 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda; -\tfrac{1}{2}x^2) \qquad (\nu > -1).$$

Using Kummer's formula

$$ {}_1F_1(\alpha, \rho; z) = e^z {}_1F_1(\rho-\alpha; \rho; -z), $$

we find that the function

$$ x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2} {}_1F_1(\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu; 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda; \tfrac{1}{2}x^2) \qquad (\nu > -1) $$

is R_λ. In other words, the function

$$ x^{\lambda+2n+\frac{1}{2}} e^{-\frac{1}{2}x^2} {}_1F_1(-n; 1+n+\lambda; \tfrac{1}{2}x^2) \qquad (\lambda+2n > -1), $$

is R_λ.

Deductions from this result:

(α) Since, when n is a positive integer,

$$ L_n^{(\alpha)}(x) = \frac{\Gamma(1+\alpha+n)}{n!\Gamma(\alpha+1)} {}_1F_1(-n; \alpha+1; x), $$

where the expression on the left-hand side denotes the generalized Laguerre polynomial of order n, it follows that the function

$$ x^{\lambda+2n+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(n+\lambda)}(\tfrac{1}{2}x^2) $$

is R_ν for positive integral values of n. This result was given by Howell (**8**).

(β) Since, when n is a positive integer,

$$ T_\mu^n(x) = \frac{(-1)^n}{n!\Gamma(1+\mu)} {}_1F_1(-n; 1+\mu; x), $$

where $T_\mu^n(x)$ is Sonine's polynomial of order n, it follows that the function

$$ x^{\lambda+2n+\frac{1}{2}} e^{-\frac{1}{2}x^2} T_{n+\lambda}^n(\tfrac{1}{2}x^2) $$

is R_λ for positive integral values of n.

(γ) If we put $n = -\lambda-\frac{1}{2}$, our function becomes

$$ x^{-\lambda-\frac{1}{2}} e^{-\frac{1}{2}x^2} {}_1F_1(\lambda+\tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}x^2). $$

As a particular case, when λ is an integer, this becomes a constant multiple of

$$ x^{-\lambda-\frac{1}{2}} e^{-\frac{1}{2}x^2} D_{2\lambda-1}(x). \qquad (3.2) $$

(δ) When $n = \frac{1}{2}-\lambda$, the function is

$$ x^{-\lambda+\frac{1}{2}} e^{-\frac{1}{2}x^2} {}_1F_1(\lambda-\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{2}x^2). $$

If λ is an integer, this is a constant multiple of

$$ x^{-\lambda+\frac{1}{2}} e^{-\frac{1}{2}x^2} D_{-2\lambda+1}(x). \qquad (3.3) $$

The functions (3.2) and (3.3) were given by Mitra (**3**).

If we take the function (c) given above as $P(x)$ we get the same R_λ function.

4. Example II

I start with the following form of Mellin-Barnes's integral (13) for $W_{k,m}(z)$:

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(s-k)\Gamma(-s-m+\frac{1}{2})\Gamma(-s+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})} z^s\,ds,$$

where $|\arg z| < \frac{3}{2}\pi$, and neither of the numbers $k \pm m + \frac{1}{2}$ is a positive integer or zero.

From this formula we get

$$z^{\frac{1}{4}(\mu+\nu-1)}e^{\frac{1}{2}z}W_{-\frac{1}{4}-\frac{1}{4}\lambda-\frac{1}{4}\mu-\frac{1}{4}\nu,\frac{1}{4}(\nu-\mu)}(\tfrac{1}{2}z^2)$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{4}(1-\mu-\nu)-i\infty}^{\frac{1}{4}(1-\mu-\nu)+i\infty} \frac{2^{\frac{1}{2}s}\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\lambda-\frac{1}{2}s)}{2^{\frac{1}{4}(s-\mu-\nu)}\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\nu)} z^{-s}\,ds, \tag{4.1}$$

where $|\arg z| < \frac{3}{4}\pi$, and neither of the numbers $\lambda+\mu$, $\lambda+\nu$ is a negative even integer or zero, and $|\mu-\nu|$ is not an integer. When $\lambda = \frac{1}{2}\mu - \frac{1}{2}\nu - 1$, a pole appears on the imaginary axis. This may be avoided by taking the integral along the line $(p-i\infty, p+i\infty)$, where $p > 0$.

The integral in (4.1) is of the same form as in (2.3) with

$$\chi(s) = \frac{2^{\frac{1}{4}(\mu+\nu-s)}}{\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\nu)}$$

and

$$-1 < \mu+\nu < 1.$$

So the function on the left-hand side of (4.1) may be taken as the function $f(x)$ in the above theorem. Hence, taking the function (b) given in § 3 as $P(x)$ and the function on the left-hand side of (4.1) as $f(x)$, we get

$$g(x) = \int_0^\infty (xy)^{\frac{1}{4}(\nu-\mu+1)}J_{\frac{1}{4}(\mu+\nu)}(xy)y^{\frac{1}{4}(\mu+\nu-1)}e^{\frac{1}{2}y^2}\times$$

$$\times W_{-\frac{1}{4}-\frac{1}{4}\lambda-\frac{1}{4}\mu-\frac{1}{4}\nu,\frac{1}{4}(\nu-\mu)}(\tfrac{1}{2}y^2)\,dy$$

$$= x^{\frac{1}{4}(\nu-\mu+1)}\int_0^\infty y^\nu J_{\frac{1}{4}(\mu+\nu)}(xy)e^{\frac{1}{2}y^2}W_{-\frac{1}{4}-\frac{1}{4}\lambda-\frac{1}{4}\mu-\frac{1}{4}\nu,\frac{1}{4}(\nu-\mu)}(\tfrac{1}{2}y^2)\,dy.$$

Evaluating this integral by a formula given by Erdélyi (4), we get

$$g(x) = x^{\frac{1}{4}(\nu-\mu+1)}x^{\frac{1}{4}\mu+\frac{1}{4}\lambda-1}e^{\frac{1}{2}x^2}W_{-\frac{1}{4}-\frac{1}{4}\nu-\frac{1}{4}\lambda,\frac{1}{4}(\nu-\lambda)}(\tfrac{1}{2}x^2)$$

$$= x^{\frac{1}{4}(\lambda+\nu-1)}e^{\frac{1}{2}x^2}W_{-\frac{1}{4}-\frac{1}{4}\nu-\frac{1}{4}\lambda,\frac{1}{4}(\nu-\lambda)}(\tfrac{1}{2}x^2),$$

where $\mu+\nu > -2$, $\nu > -1$, $\frac{1}{2}\nu-\frac{1}{2}\mu-\lambda-1 < -\frac{1}{2}$. It follows that the function

$$x^{\lambda+2m-\frac{1}{2}}e^{\frac{1}{2}x^2}W_{-\frac{1}{2}-\lambda-3m,m}(\tfrac{1}{2}x^2), \tag{4.2}$$

where $\lambda+4m > -1$, $\mu > -1$, $4m > \lambda$, is R_λ.

This function was given by Bailey (**1**).*

If we take the function (*c*) as our $P(x)$, we get the same R_λ function.

5. In 1932 I proved (**10**) that:

If $L(x)$ is R_μ and $K(x)$ is R_ν, the function

$$Q(x) = \int_0^\infty \frac{1}{y}K(y)L\left(\frac{x}{y}\right)dy = \int_0^\infty \frac{1}{y}K\left(\frac{x}{y}\right)L(y)\,dy$$

is of the form (2.1).

If we put this function for $P(x)$ in the above theorem, we arrive at the result:

If $L(x)$ is R_μ and $K(x)$ is R_ν, and $f(x)$ is given by (2.3), *then the function*

$$g(x) = \int_0^\infty f(y)\,dy \int_0^\infty \frac{1}{u}L(u)K\left(\frac{xy}{u}\right)du$$

$$= \int_0^\infty f(y)\,dy \int_0^\infty \frac{1}{u}L\left(\frac{xy}{u}\right)K(u)\,du$$

is R_λ.

If in Theorem I we take $\lambda = \nu$, (2.3) shows that $f(x)$ becomes R_μ. For example, if in (4.1) we put $\lambda = \nu$, the function on the left-hand side becomes

$$z^{\frac{1}{2}(\mu+\nu-1)}e^{\frac{1}{2}z^2}W_{-\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu,\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}z^2) = z^{\mu+2m-\frac{1}{2}}e^{\frac{1}{2}z^2}W_{-\frac{1}{2}-\mu-3m,m}(\tfrac{1}{2}z^2),$$

which is the same function as (4.2) with μ written for λ.

If we put $\lambda = \nu$ in (3.1), the left-hand side becomes

$$x^{\frac{1}{2}(\mu+\nu-1)}e^{-\frac{1}{2}x^2}W_{\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{1}{2}(\mu-\nu)}(\tfrac{1}{2}x^2).$$

Using the well-known formula

$$W_{m+\frac{1}{2},m}(z) = z^{m+\frac{1}{2}}e^{-\frac{1}{2}z},$$

we find that the above function becomes the familiar R_μ function

$$x^{\mu+\frac{1}{2}}e^{-\frac{1}{2}x^2}.$$

* Recently, Dhar (**3**) has given another proof of the fact that the function (4.2) is R_λ. The other R_λ function which he gives,

$$x^{\lambda-2m-\frac{1}{2}}e^{\frac{1}{2}x^2}W_{-\frac{1}{2}-\lambda+3m,m}(\tfrac{1}{2}x^2),$$

is the same as (4.2) with '$-m$' written for 'm'.

Thus, when $\lambda = \nu$, Theorem I assumes the simpler form:

If $f(x)$ is R_μ, and $P(x)$ is given by (2.1), *the function*

$$g(x) = \int_0^\infty P(xy) f(y)\, dy$$

is R_ν.

This theorem was proved by me in 1932 (**9**).

It is interesting to compare Theorem I with a similar theorem proved recently by the present writer and R. V. Shastry (**11**). That theorem may be thrown into the following form.

THEOREM II. *If*

$$P(x) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\omega(s)x^{-s}\, ds, \quad (5.1)$$

where $\omega(s)$ satisfies (1.2) *in the strip* (1.3), *and*

$$f(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} 2^{is}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)x^{-s}\, ds,$$

where $\chi(s)$ satisfies (1.2) *in the same strip, then the function*

$$g(x) = \int_0^\infty P(y) f(xy)\, dy$$

is R_λ.

If $\lambda = \nu$, $f(x)$ becomes R_μ, and the theorem assumes the simpler form:

If $f(x)$ is R_μ and $P(x)$ is given by (5.1), *the function*

$$g(x) = \int_0^\infty P(y) f(xy)\, dy$$

is R_ν.

This theorem was also given by me in 1932 (**9**).

6. THEOREMS III–IV. *Theorems* I–II *remain true if $\omega(s)$ and $\chi(s)$ satisfy the equation*

$$\psi(s) = -\psi(1-s) \quad (6.1)$$

in place of (1.2).

THEOREMS V–VI. *If one of the functions $\omega(s)$, $\chi(s)$ satisfies* (1.2) *and the other satisfies* (6.1) *the functions $g(x)$ in Theorems* I–II *are $-R_\nu$.*

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ON THE EXPANSION OF A FUNCTION IN A GENERALIZED NEUMANN SERIES

By D. P. BANERJEE (*Mymensingh, Bengal*)

[Received 20 July 1938]

In expanding an arbitrary function in a generalized Neumann series* of products or squares of Bessel functions, it is difficult to obtain the expressions for the coefficients in the expansion. In this paper I show that the problem can be solved in certain cases with the help of the operational calculus. My sincere thanks are due to the referee for his kind and helpful criticisms.

I shall need the following results:†

$$\int_0^\infty xe^{-px^2}J_n(\lambda x)I_n(\mu x)\,dx = \frac{1}{2p}e^{-(\lambda^2-\mu^2)/4p}J_n\left(\frac{\lambda\mu}{2p}\right) \tag{1}$$

$$(\Re(p) > 0;\ \Re(n) > -1),$$

$$\int_0^\infty x^{n+1}e^{-px^2}J_n(\lambda x)\,dx = \frac{\lambda^n}{(2p)^{n+1}}e^{-\lambda^2/4p} \quad (\Re(p) > 0;\ \Re(n) > -1), \tag{2}$$

$$x^n = 2^n\sum_{s=0}^\infty \frac{(n+2s)(n+s-1)!}{s!}J_{n+2s}(x). \tag{3}$$

I shall also need the following lemma:‡

Lemma. *Let* $f(z) = \sum_{n=0}^\infty b_n z^n$ *be an integral function of exponential type, so that* $f(z) = O(e^{c|z|})$ *for some* c, *and let* $\phi(w) = \sum_{n=0}^\infty n!\,b_n w^{-n-1}$. *Then the series for* $\phi(w)$ *is uniformly convergent for sufficiently large* w, *i.e. for* $|w| \geqslant M+\delta > M > c$, *and, if* Γ *is a closed contour surrounding the origin and lying outside the circle* $|z| = M$, *then*

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \phi(w)e^{zw}\,dw.$$

Also, if $\Re(w) > c$,

$$\phi(w) = \int_0^\infty f(x)e^{-wx}\,dx.$$

* Watson, *Theory of Bessel Functions* (Cambridge, 1922), 524.

† Gray, Mathews, and MacRobert, *Theory of Bessel Functions* (Macmillan, 1922), 33 and 69.

‡ For the proof see Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), § 1.30.

Case I. Let $f(z)$ be an even integral function such that $|f(z)| < Ke^{c|z|^2}$ and suppose that it is to be expanded in a series of the form

$$\sum_{n=0}^{\infty} a_n J_n(\lambda z) I_n(\mu z).$$

Let $$\phi(w) = \int_0^{\infty} xe^{-wx^2} f(x)\, dx \quad (\Re(w) > c).$$

Then the analytic continuation of this function will be regular for $|w| > M$ (say) and we shall have, by the lemma,

$$f(x^{\frac{1}{2}}) = \frac{1}{2\pi i} \int_{\Gamma} 2e^{wx}\phi(w)\, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} w^{-1} e^{wx-(\lambda^2-\mu^2)/4w} 2we^{(\lambda^2-\mu^2)/4w}\phi(w)\, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} w^{-1} e^{wx-(\lambda^2-\mu^2)/4w} F\left(\frac{\lambda\mu}{2w}\right) dw, \tag{4}$$

where $$F(z) = \frac{\lambda\mu}{z} e^{[(\lambda^2-\mu^2)/2\lambda\mu]z} \phi\left(\frac{\lambda\mu}{2z}\right). \tag{5}$$

But $w\phi(w)$ is analytic for $|w| > M$; thus $F(z)$ is analytic for $|z| < \dfrac{|\lambda\mu|}{2M}$, and within this region

$$F(z) = \sum_{n=0}^{\infty} a_n J_n(z), \tag{6}$$

where*

$$a_0 = F(0), \qquad a_n = n \sum_{m=0}^{\leqslant [\frac{1}{2}n]} \frac{F^{(n-2m)}(0)}{(n-2m)!} 2^{n-2m} \frac{(n-m-1)}{m!} \quad (n \geqslant 1), \tag{7}$$

the series being uniformly convergent for $|z| \leqslant \dfrac{|\lambda\mu|}{2M} - \delta$ $(\delta > 0)$.† It follows that

$$F\left(\frac{\lambda\mu}{2w}\right) = \sum_{n=0}^{\infty} a_n J_n\left(\frac{\lambda\mu}{2w}\right),$$

the series converging uniformly on Γ. Since $w^{-1}e^{wx-(\lambda^2-\mu^2)/4w}$ is bounded on Γ, we can substitute for $F(\lambda\mu/2w)$ in (4) and integrate term by term, giving

$$f(x^{\frac{1}{2}}) = \sum_{n=0}^{\infty} \frac{a_n}{2\pi i} \int_{\Gamma} e^{wx-(\lambda^2-\mu^2)/4w} J_n\left(\frac{\lambda\mu}{2w}\right) \frac{dw}{w}$$

$$= \sum_{n=0}^{\infty} a_n J_n(\lambda x^{\frac{1}{2}}) I_n(\mu x^{\frac{1}{2}}),$$

* Watson, loc. cit., § 16.11 (3).

† Gray, loc. cit., p. 35.

the last result being obtained from equation (1) by applying the lemma to the integral function $J_n(\lambda x^{\frac{1}{2}})I_n(\mu x^{\frac{1}{2}})$. Hence we have finally

$$f(x) = \sum_{n=0}^{\infty} a_n J_n(\lambda x) I_n(\mu x),$$

where a_n is given by (7).

Example 1. Let $f(x) = x^n J_n\{(\lambda^2-\mu^2)^{\frac{1}{2}}x\}$. Then

$$\begin{aligned}\phi(w) &= \int_0^{\infty} e^{-wx^2} x^{n+1} J_n\{(\lambda^2-\mu^2)^{\frac{1}{2}}x\}\, dx \\ &= (\lambda^2-\mu^2)^{n/2}(2w)^{-n-1}e^{-(\lambda^2-\mu^2)/4w},\end{aligned}$$

by (2). Substituting into (5) we have

$$\begin{aligned}F(z) &= \left(\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\right)^{n/2} z^n \\ &= 2^n\left(\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\right)^{n/2} \sum_{s=0}^{\infty} \frac{(n+2s)(n+s-1)!}{s!} J_{n+2s}(z),\end{aligned}$$

by (3), and thus it follows from (6) and (7) that

$$\begin{aligned}&x^n J_n\{(\lambda^2-\mu^2)^{\frac{1}{2}}x\} \\ &\quad = 2^n\left(\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\right)^{n/2} \sum_{s=0}^{\infty} \frac{(n+2s)(n+s-1)!}{s!} J_{n+2s}(\lambda x) I_{n+2s}(\mu x). \quad (8)\end{aligned}$$

Example 2. Putting $i\mu$ for μ in (8), we have

$$\begin{aligned}&x^n J_n\{(\lambda^2+\mu^2)^{\frac{1}{2}}x\} \\ &\quad = 2^n\left(\frac{\lambda^2+\mu^2}{\lambda^2\mu^2}\right)^{n/2} \sum_{s=0}^{\infty} \frac{(-1)^s(n+2s)(n+s-1)!}{s!} J_{n+2s}(\lambda x) J_{n+2s}(\mu x).\end{aligned}$$

Case II. Suppose that $f(x)$ satisfies the same conditions as before and that it is to be expanded in the form $\sum_{n=0}^{\infty} a_n x^n J_n(x)$. We have

$$\begin{aligned}f(x^{\frac{1}{2}}) &= \frac{1}{2\pi i}\int_{\Gamma} 2e^{wx}\phi(w)\, dw \\ &= \frac{1}{2\pi i}\int_{\Gamma} w^{-1}e^{wx-1/4w} 2we^{1/4w}\phi(w)\, dw \\ &= \frac{1}{2\pi i}\int_{\Gamma} w^{-1}e^{wx-1/4w} F(1/2w)\, dw,\end{aligned}$$

where $F(z) = z^{-1}e^{\frac{1}{2}z}\phi\left(\frac{1}{2z}\right)$.

Since $F(z)$ is analytic for sufficiently small z, we have

$$f(x^{\frac{1}{2}}) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \frac{1}{\pi i} \int_{\Gamma} e^{wx - 1/4w} (2w)^{-n-1} \, dw$$

$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^{n/2} J_n(x^{\frac{1}{2}}),$$

by the formula reciprocal to (2) (with $\lambda = 1$). Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n J_n(x).$$

The analysis is easily justified as in case I.

Example 3. Let $f(x) = x^{2m}$. Then $\phi(w) = \frac{1}{2} m! \, w^{-m-1}$ and

$$F(z) = m!(2z)^m e^{z/2} = \sum_{n=m}^{\infty} \frac{m!}{(n-m)!} 2^{2m} \left(\frac{z}{2}\right)^n.$$

It follows that $$x^{2m} = 2^{2m} \sum_{n=m}^{\infty} \frac{m!}{(n-m)!} \left(\frac{x}{2}\right)^n J_n(x).$$

Case III. When case II fails, or when the evaluation of $\phi(w)$ is difficult, we may sometimes proceed as follows. Let $f(z)$ be an even integral function such that $|f(z)| < Ke^{c|z|}$, and let $\phi(w)$ be defined as in the lemma. Since $b_n = 0$ (n odd), $w\phi(w)$ is a function of w^2. We have

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} e^{wx} \phi(w) \, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{wx}}{(1+w^2)^{\frac{1}{2}}} (1+w^2)^{\frac{1}{2}} \phi(w) \, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{wx}}{(1+w^2)^{\frac{1}{2}}} F\left(\frac{1}{1+w^2}\right) dw,$$

where $$F\left(\frac{1}{1+w^2}\right) = (1+w^2)^{\frac{1}{2}} \phi(w) = \left(1 + \frac{1}{w^2}\right)^{\frac{1}{2}} w\phi(w).$$

This is an analytic function of $\dfrac{1}{1+w^2}$ for sufficiently large w; hence

$$f(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{wx}}{(1+w^2)^{n+\frac{1}{2}}} \, dw$$

$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \frac{\pi^{\frac{1}{2}}}{2^n \Gamma(n+\frac{1}{2})} x^n J_n(x),$$

since $x^n J_n(x)$ and $\frac{2^n \Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}(1+w^2)^{n+\frac{1}{2}}}$ are Laplace transforms.*

Example 4. If $f(x) = \cos x$, then $\phi(w) = w/(1+w^2)$ and

$$F\left(\frac{1}{1+w^2}\right) = \frac{w}{(1+w^2)^{\frac{1}{2}}} = 1 - \sum_{n=1}^{\infty} \frac{1.3\ldots(2n-3)}{2^n n!(1+w^2)^n}.$$

Hence
$$\cos x = 1 - \sum_{n=1}^{\infty} \frac{(\frac{1}{2}x)^n J_n(x)}{n!\,(2n-1)}.$$

Example 5. If $f(x) = x \sin x$, then $\phi(w) = 2w/(1+w^2)^2$ and

$$F\left(\frac{1}{1+w^2}\right) = \frac{2w}{(1+w^2)^{\frac{3}{2}}} = 2\left\{\frac{1}{1+w^2} - \sum_{n=1}^{\infty} \frac{1.3\ldots(2n-3)}{2^n n!(1+w^2)^{n+1}}\right\}.$$

Hence
$$x \sin x = 2\left\{xJ_1(x) - 2\sum_{n=1}^{\infty} \frac{(\frac{1}{2}x)^{n+1} J_{n+1}(x)}{n!(4n^2-1)}\right\}.$$

* Titchmarsh, loc. cit., p. 207.

THREE TRIPLE INTEGRALS

By G. N. WATSON (*Birmingham*)

[Received 3 June 1939]

The desirability of investigating the triple integrals

$$I_1 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dudvdw}{1-\cos u\cos v\cos w},$$

$$I_2 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dudvdw}{3-\cos v\cos w-\cos w\cos u-\cos u\cos v},$$

$$I_3 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dudvdw}{3-\cos u-\cos v-\cos w}$$

has arisen as a consequence of their having appeared in a recent paper* in ferromagnetic anisotropy by F. van Peype, a pupil of H. A. Kramers. The problem of evaluating them was proposed by Kramers to R. H. Fowler who communicated it to G. H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the journey to Birmingham without difficulty.

It is possible to express all three of the integrals in terms of simple surds, the number π, and certain complete elliptic integrals. The elliptic integrals which occur in I_1 and I_2 are easily expressible in terms of gamma functions whose arguments are simple fractions. This is not the case, so far as I know, with the elliptic integral occurring in I_3; though this elliptic integral is quite familiar, since its modulus is one of the simplest of the set of singular moduli.

The value of I_1 appears to be fairly well known, but, for the sake of completeness, I give two methods of determining it. So far as I know, no method of evaluating I_2 and I_3 has hitherto been discovered; my methods involve, in the one case, the use of a certain expansion for the square of a complete elliptic integral and, in the other case, the use of a formula discovered by Bailey connecting the product

* *Physica*, 5 (1938), 465.

of two hypergeometric functions with a hypergeometric function of two variables of Appell's fourth type.

The results which I have obtained are as follows:

$$I_1 = \frac{4K_0^2}{\pi^2} = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3},$$

$$I_2 = \frac{K_1^2\sqrt{3}}{\pi^2} = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4},$$

$$I_3 = (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})(2K_2/\pi)^2,$$

where K_0, K_1, K_2 denote the complete elliptic integrals with moduli

$$\sin 45^\circ, \qquad \sin 15^\circ, \qquad (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$$

respectively. From these results it is easy to compute the numerical values of I_1, I_2, and I_3, namely

$$I_1 = 1{\cdot}39320 \quad 39297,$$

$$I_2 = 0{\cdot}44822 \quad 03944,$$

$$I_3 = 0{\cdot}50546 \quad 20197.$$

I start with all three integrals in the same way, by writing

$$\tan\tfrac{1}{2}u = x, \qquad \tan\tfrac{1}{2}v = y, \qquad \tan\tfrac{1}{2}w = z,$$

and then, regarding x, y, z as Cartesian coordinates, I change to polar coordinates by the transformation

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta;$$

after writing $2\phi = \psi$, I next bisect the ψ-range of integration; the integrands at corresponding points in the two parts of the range being equal, this procedure gives

$$\int_0^{\frac{1}{2}\pi} (\ldots)\,d\phi = \tfrac{1}{2}\int_0^{\pi} (\ldots)\,d\psi = \int_0^{\frac{1}{2}\pi} (\ldots)\,d\psi.$$

There is no difficulty in justifying the various changes which are made in the order of carrying out the integrations, because all the integrands are positive throughout and the triple integrals are transformed into integrals which are evidently convergent.

By treating I_1 in the manner just described, we get

$$I_1 = \frac{8}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{(1+x^2)(1+y^2)(1+z^2)-(1-x^2)(1-y^2)(1-z^2)}$$

$$= \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{x^2+y^2+z^2+x^2y^2z^2}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\phi}{1+r^4\sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\psi}{1+\frac{1}{4}r^4\sin^4\theta\cos^2\theta\sin^2\psi}.$$

In the innermost integral replace the variable r by a new variable t defined by the formula

$$t = r\sin\theta\sqrt{(\tfrac{1}{2}\cos\theta\sin\psi)};$$

this transformation gives

$$I_1 = \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\theta d\psi}{(1+t^4)\sqrt{(\frac{1}{2}\cos\theta\sin\psi)}}$$

$$= \frac{4\sqrt{2}}{\pi^3}\int_0^\infty \frac{dt}{1+t^4}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\cos\theta}}\int_0^{\frac{1}{2}\pi}\frac{d\psi}{\sqrt{\sin\psi}}.$$

Now it is an easy exercise in contour integration to prove that

$$\int_0^\infty \frac{dt}{1+t^4} = \frac{\pi}{2\sqrt{2}},$$

and it follows from known properties of the first Eulerian integral that

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\cos\theta}} = \int_0^{\frac{1}{2}\pi}\frac{d\psi}{\sqrt{\sin\psi}} = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{(2\pi)}}.$$

By combining these results we immediately get

$$I_1 = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3},$$

which gives one of the values stated for I_1.

The second method of dealing with I_1 is to effect one of the integrations forthwith and get

$$I_1 = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{dvdw}{\sqrt{(1-\cos^2 v\cos^2 w)}}.$$

We now write

$$a_0 = 1, \qquad a_n = \frac{1.3.5...(2n-1)}{2.4.6...(2n)} \quad (n \geqslant 1)$$

for brevity, expand the integrand in ascending powers of $\cos^2 v\cos^2 w$ and integrate term by term; we thus get

$$I_1 = \frac{1}{\pi^2}\sum_{n=0}^{\infty} a_n \int_0^\pi\int_0^\pi \cos^{2n}v\cos^{2n}w\, dvdw = \sum_{n=0}^{\infty} a_n^3,$$

the term-by-term integration being justified by the convergence of the series on the right.

Now it is known* that, when $0 \leqslant k \leqslant \sin 45°$,

$$K^2 = \tfrac{1}{4}\pi^2 \sum_{n=0}^{\infty} a_n^3 (2kk')^{2n};$$

and so, taking $k = \sin 45°$, we find that

$$I_1 = \frac{4K_0^2}{\pi^2},$$

which gives the other value stated for I_1.

The numerical value of I_1 is now immediately derivable from results contained in Legendre's tables of elliptic integrals and gamma-functions.

We now turn to I_2. When we transform it in the prescribed manner, we get

$$\begin{aligned} I_2 &= \frac{8}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{3\prod(1+x^2) - \sum(1-y^2)(1-z^2)(1+x^2)} \\ &= \frac{2}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{x^2+y^2+z^2+y^2z^2+z^2x^2+x^2y^2} \\ &= \frac{2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\phi}{1+r^2\sin^2\theta(\cos^2\theta+\sin^2\theta\sin^2\phi\cos^2\phi)} \end{aligned}$$

* Expansions of this kind have been discussed by various writers; I dealt with them myself many years ago at some length, *Quart. J. of Math.* **39** (1908), 27–51.

$$= \frac{2}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\infty} \frac{\sin\theta \, dr d\theta d\psi}{1+r^2 \sin^2\theta(\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2\psi)}$$

$$= \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta d\psi}{\sqrt{(\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2\psi)}}$$

$$= \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\infty} \frac{dt d\psi}{\sqrt{\{(1+t^2)(1+\frac{1}{4}t^2\sin^2\psi)\}}},$$

where, at the last step, $\tan\theta$ has been replaced by t.

We have now to introduce complete elliptic integrals of the first kind in which it is convenient to make explicit mention of the modulus. We therefore write $K(k)$ to denote the complete elliptic integral of the first kind with modulus k and we write $K'(k)$ to denote the complete elliptic integral of the first kind in which the modulus k' is the modulus complementary to k, so that $K'(k) = K(k')$.

Since
$$u = \int_0^{\mathrm{sc}(u,k)} \frac{dt}{\sqrt{\{(1+t^2)(1+k'^2t^2)\}}},$$

we have
$$\int_0^{\infty} \frac{dt}{\sqrt{\{(1+t^2)(1+k'^2t^2)\}}} = K(k) = K'(k'),$$

whence we obtain the preliminary result

$$I_2 = \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} K'(\tfrac{1}{2}\sin\psi) \, d\psi.$$

Now, with the meaning already assigned to a_n, we have

$$K'(k) = \sum_{n=0}^{\infty} a_n^2 \{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log k\} k^{2n};$$

and, for values of k between 0 and 1, every term in the series on the right is positive. We consequently have

$$I_2 = \frac{1}{\pi^2} \sum_{n=0}^{\infty} a_n^2 \int_0^{\frac{1}{2}\pi} \{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log(\tfrac{1}{2}\sin\psi)\}(\tfrac{1}{2}\sin\psi)^{2n} \, d\psi,$$

term-by-term integration being permissible because the series on the

right proves to be convergent. To evaluate the general term on the right, we observe that

$$a_n^2\{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log k\}k^{2n} = -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)}{\Gamma(n+1+\epsilon)}k^{2n+\epsilon}\right)\right]_{\epsilon=0},$$

and so we have

$$\begin{aligned}
a_n^2\int_0^{\frac{1}{2}\pi}&\{\psi(n+1)-\psi(n+\tfrac{1}{2})-\log(\tfrac{1}{2}\sin\psi)\}(\tfrac{1}{2}\sin\psi)^{2n}\,d\psi\\
&= -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)}{\Gamma(n+1+\epsilon)}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\sin\psi)^{2n+\epsilon}\,d\psi\right)\right]_{\epsilon=0}\\
&= -\frac{a_n}{\sqrt{\pi}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\epsilon)\Gamma(n+\frac{1}{2}+\frac{1}{2}\epsilon)\sqrt{\pi}}{2^{2n+\epsilon+1}\Gamma(n+1+\epsilon)\Gamma(n+1+\frac{1}{2}\epsilon)}\right)\right]_{\epsilon=0}\\
&= \tfrac{1}{4}\pi a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})+2\log 2\}(\tfrac{1}{2})^{2n},
\end{aligned}$$

the interchange of the order of differentiation and integration presenting no particular theoretical difficulty. We have consequently arrived at the result

$$I_2 = \frac{1}{4\pi}\sum_{n=0}^{\infty} a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})+2\log 2\}(\tfrac{1}{2})^{2n}.$$

Now in my previously cited paper it is proved that, for

$$0 \leqslant k \leqslant \sin 45^\circ$$

(and by symmetry also for $\sin 45^\circ \leqslant k \leqslant 1$), we have

$$K(k)K(k') = \tfrac{1}{4}\pi\sum_{n=0}^{\infty} a_n^3\{3\psi(n+1)-3\psi(n+\tfrac{1}{2})-2\log(2kk')\}(2kk')^{2n}.$$

When we compare this expansion with the formula just obtained for I_2, it is clear that

$$I_2 = \frac{K(k_1)K(k_1')}{\pi^2},$$

where k_1 is the modulus between 0 and $\sin 45^\circ$ which satisfies the condition $2k_1k_1' = \frac{1}{2}$, so that $k_1 = \sin 15^\circ$ and $k_1' = \sin 75^\circ$. If we now use Legendre's well-known result that $K(\sin 75^\circ) = K(\sin 15^\circ)\sqrt{3}$ together with his formula expressing $K(\sin 15^\circ)$ in terms of gamma-functions, we immediately find that

$$I_2 = \frac{K^2(\sin 15^\circ)\sqrt{3}}{\pi^2} = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4};$$

these are the formulae for I_2 stated at the beginning of this paper,

and the computation of the numerical value of I_2 is merely a matter of employing a table of gamma-functions or of complete elliptic integrals.

We finally consider I_3. When we transform it in the prescribed manner, we get

$$I_3 = \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{dxdydz}{\sum x^2+2\sum y^2z^2+3x^2y^2z^2}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\phi}{1+2r^2\sin^2\theta(\cos^2\theta+\sin^2\theta\sin^2\phi\cos^2\phi)+ +3r^4\sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi}$$

$$= \frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{\sin\theta\, drd\theta d\psi}{1+2r^2\sin^2\theta(\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2\psi)+\frac{3}{4}r^4\sin^4\theta\cos^2\theta\sin^2\psi}.$$

Now replace r by a new variable t defined by the equation $r\sin\theta = t\sqrt{2}$ and then perform the integration with respect to θ. This procedure gives

$$I_3 = \frac{4\sqrt{2}}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\theta d\psi}{1+t^2(4\cos^2\theta+\sin^2\theta\sin^2\psi)+3t^4\cos^2\theta\sin^2\psi}$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^\infty \frac{dtd\psi}{\sqrt{\{(1+4t^2+3t^4\sin^2\psi)(1+t^2\sin^2\psi)\}}}.$$

Next take $\tan\psi$ as a new variable ξ, and then write $\xi = \zeta/\sqrt{(1+t^2)}$ and perform the integration with respect to t. We thus get

$$I_3 = \frac{2\sqrt{2}}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\xi dt}{\sqrt{\{(1+\xi^2+4t^2+4t^2\xi^2+3t^4\xi^2)(1+\xi^2+t^2\xi^2)\}}}$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\zeta dt}{\sqrt{\{(1+t^2)(1+4t^2+\zeta^2+3t^2\zeta^2)(1+\zeta^2)\}}}$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^\infty K'\left(\sqrt{\frac{1+\zeta^2}{4+3\zeta^2}}\right)\frac{d\zeta}{\sqrt{\{(1+\zeta^2)(4+3\zeta^2)\}}}$$

$$= \frac{2\sqrt{2}}{\pi^2}\int_0^{\frac{1}{2}\pi} K'\left(\frac{1}{\sqrt{(4-\sin^2\chi)}}\right)\frac{d\chi}{\sqrt{(4-\sin^2\chi)}},$$

by writing $\zeta = \tan\chi$.

We now replace $K'(k)$ by a differential coefficient, as in the corresponding discussion of I_2. The expression which it is now convenient to take is not the one used in dealing with I_2 but

$$2\pi K'(k) = -\left[\frac{d}{d\epsilon}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)k^{2n+2\epsilon}}{n!\,\Gamma(n+1+2\epsilon)}\right]_{\epsilon=0}.$$

The changes of order of integration, differentiation, and summation are even more easy to justify than in the case of I_2, since the range of values of k now under consideration extends only from $\frac{1}{2}$ to $1/\sqrt{3}$ and it has not now got 0 or 1 for an end-point. We thus find that

$$I_3 = -\frac{\sqrt{2}}{\pi^3}\left[\frac{d}{d\epsilon}\int_0^{\frac{1}{2}\pi}\left(\sum_{n=0}^{\infty}\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\frac{1}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)d\chi\right]_{\epsilon=0}$$

$$= -\frac{\sqrt{2}}{\pi^3}\left[\frac{d}{d\epsilon}\sum_{n=0}^{\infty}\left(\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\int_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)\right]_{\epsilon=0}.$$

The next step is the integration of the various powers of $4-\sin^2\chi$. We define c to be the smaller root of the quadratic equation

$$c^2+1 = 14c,$$

so that $c = 7-4\sqrt{3}$, and we then have

$$\begin{aligned} 4c(4-\sin^2\chi) &= 2c(7+\cos 2\chi) \\ &= 1+2c\cos 2\chi+c^2 \\ &= (1+ce^{2i\chi})(1+ce^{-2i\chi}), \end{aligned}$$

whence it follows that

$$\frac{1}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}} = \frac{(4c)^{n+\frac{1}{2}+\epsilon}}{\{(1+ce^{2i\chi})(1+ce^{-2i\chi})\}^{n+\frac{1}{2}+\epsilon}}.$$

We now expand each factor in the denominator on the right in a series of ascending powers of c, multiply the two series together, and integrate. Since the integrals of the products of unequal powers of $e^{2i\chi}$ and $e^{-2i\chi}$ are all zero, this procedure gives immediately

$$\int_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}} = \tfrac{1}{2}\pi(4c)^{n+\frac{1}{2}+\epsilon}\,{}_2F_1(n+\tfrac{1}{2}+\epsilon, n+\tfrac{1}{2}+\epsilon; 1; c^2),$$

whence it follows that

$$\sum_{n=0}^{\infty}\left(\frac{\Gamma^2(n+\frac{1}{2}+\epsilon)}{n!\,\Gamma(n+1+2\epsilon)}\int_0^{\frac{1}{2}\pi}\frac{d\chi}{(4-\sin^2\chi)^{n+\frac{1}{2}+\epsilon}}\right)$$

$$=\tfrac{1}{2}\pi(4c)^{\frac{1}{2}+\epsilon}\frac{\Gamma^2(\frac{1}{2}+\epsilon)}{\Gamma(1+2\epsilon)}\mathfrak{F}_4(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon,1;4c,c^2),$$

where $\mathfrak{F}_4$ denotes the fourth kind of Appell's hypergeometric function of two variables.

We now avail ourselves of Bailey's result* that

$$\mathfrak{F}_4(\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;a(1-b),b(1-a))$$

$$={}_2F_1(\alpha,\beta;\gamma;a)\,{}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;b),$$

which, for a given value of the (complex) number a whose modulus is less than unity, is valid for values of b in the domain surrounding the origin for which the condition

$$\sqrt{|a(1-b)|}+\sqrt{|b(1-a)|}<1$$

is satisfied. When

$$a(1-b)=4c=28-16\sqrt{3},$$

$$b(1-a)=c^2=97-56\sqrt{3},$$

we have $$\sqrt{|a(1-b)|}+\sqrt{|b(1-a)|}-1=10-6\sqrt{3}<0,$$

and so the requisite condition is satisfied. Moreover, when a has any given positive value between 0 and 1, the domain of values of b surrounding the origin within which the condition is satisfied contains that portion of the real axis which extends from the origin to $1-a$. We therefore must select as the values of a and b that pair of solutions of the equations

$$a(1-b)=28-16\sqrt{3},$$

$$b(1-a)=97-56\sqrt{3}$$

which satisfy the condition $a+b<1$, assuming such a pair to exist. There is, however, no difficulty in verifying that

$$a=24\sqrt{2}-14\sqrt{6}+20\sqrt{3}-34,$$

$$b=24\sqrt{2}-14\sqrt{6}-20\sqrt{3}+35$$

are solutions of the simultaneous equations such that

$$a+b-1=\sqrt{4608}-\sqrt{4704}<0.$$

* W. N. Bailey, *Quart. J. of Math.* (Oxford), 4 (1933), 305–8.

We now write

$$k_2^2 \equiv b \equiv (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2,$$
$$k_3^2 \equiv 1-a \equiv (2-\sqrt{3})^2(\sqrt{3}+\sqrt{2})^2,$$

and we immediately find that

$$I_3 = -\frac{1}{\pi^2\sqrt{2}}\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)(k_2'k_3')^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\times\right.\right.$$
$$\left.\left.\times F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1;k_2^2)\right)\right]_{\epsilon=0}.$$

An application of Euler's formula to the second hypergeometric function on the right gives

$$F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1;k_2^2) = (1-k_2^2)^{-\frac{1}{2}-\epsilon}F(\tfrac{1}{2}-\epsilon,\tfrac{1}{2}+\epsilon;1;-k_2^2/k_2'^2);$$

now $F(\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon;1;-k_2^2/k_2'^2)$ is an even function of ϵ, and so its differential coefficient with respect to ϵ obviously vanishes with ϵ. Consequently, when we differentiate the product of the functions

$$F(\tfrac{1}{2}-\epsilon,\tfrac{1}{2}+\epsilon;1;-k_2^2/k_2'^2),$$
$$\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2),$$

we immediately obtain the formula

$$I_3 = -\frac{1}{\pi^2\sqrt{2}}F(\tfrac{1}{2},\tfrac{1}{2};1;-k_2^2/k_2'^2)\times$$
$$\times\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{1+2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\right)\right]_{\epsilon=0}.$$

A second application of Euler's formula now gives

$$I_3 = -\frac{k_2'k_3'}{\pi^2\sqrt{2}}F(\tfrac{1}{2},\tfrac{1}{2};1;k_2^2)\times$$
$$\times\left[\frac{d}{d\epsilon}\left(\frac{\Gamma^2(\frac{1}{2}+\epsilon)k_3'^{2\epsilon}}{\Gamma(1+2\epsilon)}F(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon;1+2\epsilon;k_3'^2)\right)\right]_{\epsilon=0}$$
$$= \frac{2k_2'k_3'K_2K_3\sqrt{2}}{\pi^2},$$

where K_2 and K_3 naturally denote the complete elliptic integrals with moduli k_2 and k_3.

Now, if τ_n denotes the parameter associated with the modulus k_n, that is to say that τ_n is the quotient

$$\tau_n = iK_n'/K_n,$$

where K_n' denotes the complete elliptic integral whose modulus k_n' is complementary to k_n, it is known* that k_2 and k_3 are the moduli for which

$$\tau_2 = i\sqrt{6}, \qquad \tau_3 = i\sqrt{\tfrac{2}{3}}.$$

The quarter periods K_2 and K_3' are consequently connected by Landen's transformation, so that we have

$$K_3' = (1+k_2)K_2,$$

in addition to the previous relation

$$K_3' = K_3\sqrt{\tfrac{2}{3}}.$$

By means of these results the formula for I_3 reduces to

$$I_3 = \frac{2k_2'k_3'(1+k_2)K_2^2\sqrt{3}}{\pi^2};$$

and, when we replace the moduli by their values in surd form, we immediately find that

$$I_3 = (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})(2K_2/\pi)^2,$$

which is the value stated at the beginning of the paper. In computing the numerical value of I_3 from this result, I found it quickest to use the formula

$$2K_2/\pi = (1+2e^{-\pi\sqrt{6}}+2e^{-4\pi\sqrt{6}}+2e^{-9\pi\sqrt{6}}+\ldots)^2$$

for the preliminary computation of $2K_2/\pi = 1{\cdot}00182\ 06717\ 32$.

* See e.g. Greenhill, *Proc. London Math. Soc.* (1), 19 (1889), 351.

ON TCHEBYCHEFF POLYNOMIALS

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[Received 10 June 1939]

In §§ 1–4 I shall reproduce some definitions and some known results concerning T-polynomials.

1. We say that a system

$$\phi_0(x),\ \phi_1(x),\ldots,\ \phi_n(x)$$

of $n+1$ real functions, continuous in the closed interval (a, b), form a T-system of order n in (a, b) if no linear combination

$$P_n(x) = A_0\phi_0(x)+A_1\phi_1(x)+\ldots+A_n\phi_n(x),$$

with real coefficients, has more than n distinct zeros in this interval unless the coefficients are all zero, and we then call $P_n(x)$ a T-polynomial (or simply a polynomial) of the system.

Let $\bar{x}$ be a zero of $P_n(x)$. Then we say that $\bar{x}$ is a simple zero of $P_n(x)$, if $P_n(x)$ changes sign as x passes through $\bar{x}$. If $P_n(x)$ preserves the same sign as x passes through $\bar{x}$, we say that $\bar{x}$ is a double zero of $P_n(x)$. If $P_n(x)$ has a zero at an end-point of the interval (a, b), this zero is, of course, counted as simple. In this paper I am concerned with the problem of constructing T-polynomials with previously assigned zeros.

2. Let $x_0, x_1,\ldots, x_n$ be any $n+1$ points of (a, b). We write

$$D_n(x_0, x_1,\ldots, x_n) = \begin{vmatrix} \phi_0(x_0) & \phi_1(x_0) & . & . & . & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & . & . & . & \phi_n(x_1) \\ . & . & . & . & . & . \\ \phi_0(x_n) & \phi_1(x_n) & . & . & . & \phi_n(x_n) \end{vmatrix}.$$

THEOREM 1. *A necessary and sufficient condition for* $\phi_0(x), \phi_1(x),\ldots, \phi_n(x)$ *to form a* T*-system is that*

$$D_n(x_0, x_1,\ldots, x_n) \neq 0$$

provided only that $x_0, x_1,\ldots, x_n$ *are all distinct.* (1)

Suppose there exist $n+1$ distinct points such that

$$D_n(x_0, x_1,\ldots, x_n) = 0.$$

Then, by a theorem on determinants, there exist numbers A_0, $A_1,\ldots, A_n$, not all zero, such that

$$\sum_{k=0}^{n} A_k\phi_k(x) = 0$$

for $x = x_0, x_1, \ldots, x_n$. But this is impossible for a T-system. Hence the condition (1) is a necessary condition.

On the other hand, suppose that condition (1) is satisfied. Then, if $x_0, x_1, \ldots, x_n$ are all distinct, the only solution of the equations

$$\sum_{k=0}^{n} A_k \phi_k(x_i) = 0 \quad (i = 0, 1, \ldots, n)$$

is $A_k = 0$ $(k = 0, 1, \ldots, n)$. Hence our functions form a T-system, and the condition is a sufficient one.

3. We shall suppose from now on that the functions $\{\phi_k(x)\}$ form a T-system in the interval (a, b). Now let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points of (a, b). Then $D_n(x_0, x_1, \ldots, x_n)$ is of constant sign provided that

$$a \leqslant x_0 < x_1 < \ldots < x_n \leqslant b$$

and changes sign if we interchange the inequality between two consecutive x_i. It follows that the T-polynomial $D_n(x, x_1, x_2, \ldots, x_n)$ has n simple zeros at the points $x = x_1, x_2, \ldots, x_n$. Further, any T-polynomial having zeros at these n points is a constant multiple of $D_n(x, x_1, x_2, \ldots, x_n)$.

4. THEOREM 2. *If s is the number of simple zeros and d the number of double zeros of a T-polynomial $P_n(x)$ of our system, then*

$$s + 2d \leqslant n.$$

Let $x_1, x_2, \ldots, x_s$ be the simple zeros of $P_n(x)$ and $y_1, y_2, \ldots, y_d$ its double zeros. The sign of $P_n(x)$ in the neighbourhood of each y_j is constant and so, taking $n-s$ distinct points $x_1', x_2', \ldots, x_{n-s}'$ lying to the left of all the y_j, we can construct a T-polynomial

$$Q_n(x) = \pm D_n(x, x_1', x_2', \ldots, x_{n-s}', x_1, x_2, \ldots, x_s),$$

having n simple zeros at the points

$$x = x_1', x_2', \ldots, x_{n-s}', x_1, x_2, \ldots, x_s,$$

whose sign in the neighbourhood of each point y_j coincides with that of $P_n(x)$. Denote by 2δ the least distance between two zeros of $P_n(x)$ and write

$$M_j^+ = \overline{\text{bound}}\,\{|P_n(x)|\} \quad (y_j \leqslant x \leqslant y_j + \delta),$$

$$M_j^- = \overline{\text{bound}}\,\{|P_n(x)|\} \quad (y_j - \delta \leqslant x \leqslant y_j),$$

$$M = \min(M_j^+, M_j^-) \quad (j = 1, 2, \ldots, d),$$

$$N = \overline{\text{bound}}\,\{|Q_n(x)|\} \quad (a \leqslant x \leqslant b).$$

Choose a positive number λ such that $\lambda N < M$. Then to any point y_j there corresponds at least one zero of the polynomial $P_n(x)-\lambda Q_n(x)$ in each of the open intervals $(y_j-\delta, y_j)$ and $(y_j, y_j+\delta)$. Apart from these zeros the points $x_1, x_2,..., x_s$ are zeros of $P_n(x)-\lambda Q_n(x)$. Hence the polynomial $P_n(x)-\lambda Q_n(x)$ has at least $s+2d$ distinct zeros, and therefore

$$s+2d \leqslant n,$$

as was to be proved.

5. We now come to the main problem of the paper.

THEOREM 3. *If $s+2d = n$, and we are given $s+d$ distinct points $x_1, x_2,..., x_s; y_1, y_2,..., y_d$ such that*

$$a \leqslant x_i \leqslant b \quad (i = 1, 2,..., s), \qquad a < y_j < b \quad (j = 1, 2,..., d),$$

*then we can construct a T-polynomial of our system having simple zeros at the points $x_1, x_2,..., x_s$ and double zeros at the points $y_1, y_2,..., y_d$.**

First suppose $d = 1$. Take points β_1, γ_1 such that

$$\beta_1 < y_1 < \gamma_1$$

and, further, such that the interval (β_1, γ_1) contains none other of the given points. We can construct a unique T-polynomial taking the value unity at the point β_1 and having n simple zeros at the points $\gamma_1, x_1, x_2,..., x_s, y_1$. Denote this polynomial by $P_n^{(1)}(x)$, and denote by $Q_n^{(1)}(x)$ the unique T-polynomial taking the value unity at the point γ_1 and having simple zeros at the points $\beta_1, x_1, x_2,..., x_s, y_1$. Observe that the polynomials $P_n^{(1)}(x)$, $Q_n^{(1)}(x)$ are of different sign in the interval (β_1, γ_1) and have the same sign outside this interval.

Now let $\qquad \beta_1 \leqslant y < y_1 \quad$ or $\quad y_1 < y < \gamma_1$

and write

$$\lambda_1(y) = -\frac{Q_n^{(1)}(y)}{P_n^{(1)}(y)}.$$

Then $\lambda_1(y)$ is non-negative in the ranges of y considered. Further, the T-polynomial $Q_n^{(1)}(x)+\lambda_1(y)P_n^{(1)}(x)$ takes the value unity at the point γ_1 and has n zeros at the points $x_1, x_2,..., x_s, y_1, y$. It follows that $\lambda_1(y)$ cannot assume the same value for different values of y.

* This result has been given by S. N. Bernstein, *Extremal Properties of Polynomials* (in Russian) [1937], p. 9, lemma II. Bernstein's proof, however, does not seem to be complete, and I give here a proof on entirely different lines.

Hence, since $\lambda_1(y)$ is continuous, it is monotone increasing (in the strict sense) in the ranges of y considered, and the limits

$$\underline{\lambda}_1 = \lim_{y \to y_1 - 0} \lambda_1(y), \qquad \bar{\lambda}_1 = \lim_{y \to y_1 + 0} \lambda_1(y)$$

exist finitely and $\qquad 0 < \underline{\lambda}_1 \leqslant \bar{\lambda}_1.$

Choose μ_1 so that $\underline{\lambda}_1 \leqslant \mu_1 \leqslant \bar{\lambda}_1$ and consider the T-polynomial

$$R_n^{(1)}(x) = Q_n^{(1)}(x) + \mu_1 P_n^{(1)}(x).$$

$R_n^{(1)}(x)$ has a zero at the point y_1 and, since the function $\lambda_1(y)$ never assumes the value μ_1, it can have no other zero in the interval (β_1, γ_1). Further, $R_n^{(1)}(x)$ takes the value unity at the point γ_1 and the positive value μ_1 at the point β_1. Hence y_1 is a double zero of $R_n^{(1)}(x)$. But $R_n^{(1)}(x)$ has also zeros at the points $x_1, x_2, \ldots, x_s$ and—by Theorem 2, or from the fact that $P_n^{(1)}(x)$, $Q_n^{(1)}(x)$ are of the same sign outside the interval (β_1, γ_1)—these must be simple zeros. $R_n^{(1)}(x)$ therefore satisfies the required conditions.

Observe that the solution to the problem is unique (except for a constant multiple) if and only if $\underline{\lambda}_1 = \bar{\lambda}_1$. We shall return to this point later and show by an example that this is not always the case.

We now pass to the case $d = 2$. Take points β_2, γ_2 such that

$$\beta_2 < \underline{y}_2 < \gamma_2$$

and, further, such that the interval (β_2, γ_2) contains none other of the given points. By the preceding argument we can construct a T-polynomial $P_n^{(2)}(x)$ taking the value unity at the point β_2 and having a double zero at the point y_1 and simple zeros at the points $\gamma_2, x_1, x_2, \ldots, x_s, y_2$. Similarly we can construct a polynomial $Q_n^{(2)}(x)$ which takes the value unity at the point γ_2 and has a double zero at the point y_1 and simple zeros at the points $\beta_2, x_1, x_2, \ldots, y_2$.

Now let $\qquad \beta_2 \leqslant y < y_2 \quad \text{or} \quad y_2 < y < \gamma_2$

and write $\qquad \lambda_2(y) = -\dfrac{Q_n^{(2)}(y)}{P_n^{(2)}(y)}.$

$P_n^{(2)}(x)$ and $Q_n^{(2)}(x)$ are of different sign in the interval (β_2, γ_2) and of the same sign outside this interval. It follows that $\lambda_2(y)$ is non-negative and that the polynomial $Q_n^{(2)}(x) + \lambda_2(y) P_n^{(2)}(x)$ has a double zero at the point y_1 and simple zeros at the points $x_1, x_2, \ldots, x_s, y_2, y$. Hence, by Theorem 2, $\lambda_2(y)$ cannot assume the same value for different values of y and we deduce, exactly as before, that the limits

$$\underline{\lambda}_2 = \lim_{y \to y_2 - 0} \lambda_2(y), \qquad \bar{\lambda}_2 = \lim_{y \to y_2 + 0} \lambda_2(y)$$

exist finitely and $0 < \underline{\lambda}_2 \leqslant \bar{\lambda}_2$,

and that, if we take any number μ_2 such that $\underline{\lambda}_2 \leqslant \mu_2 \leqslant \bar{\lambda}_2$, then the T-polynomial

$$R_n^{(2)}(x) = Q_n^{(2)}(x) + \mu_2 P_n^{(2)}(x)$$

satisfies all our requirements.

The general case of the theorem follows immediately by repeating the above argument.

6. Finally, we give an example which shows that the solution to Theorem 3 is not necessarily unique.

THEOREM 4. *The functions*

$$1, \qquad |x|, \qquad \sin x$$

form a T-system of order 2 *in the interval* $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

We have to show that, if a, b, c are not all zero, then the function

$$f(x) = a + b|x| + c\sin x$$

cannot have more than two zeros in the given interval. This is evidently true if $c = 0$. Suppose, then, that c is different from zero. Without loss of generality we may suppose $c = 1$. Then

$$\begin{aligned} f'(x) &= b + \cos x \qquad (x > 0), \\ f'(+0) &= b + 1, \\ f'(-0) &= -b + 1, \\ f'(x) &= -b + \cos x \qquad (x < 0). \end{aligned}$$

Suppose $b > 0$. Then, if $b \leqslant 1$, $f(x)$ is strictly decreasing in the interval $(-\frac{1}{2}\pi, -\arccos b)$ and strictly increasing in $(-\arccos b, \frac{1}{2}\pi)$; while, if $b > 1$, $f(x)$ is strictly decreasing in $(-\frac{1}{2}\pi, 0)$ and strictly increasing in $(0, \frac{1}{2}\pi)$. Similar arguments apply if $b < 0$, and in either case $f(x)$ cannot have more than two zeros in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. If $b = 0$, then $f(x)$ can have at most one zero in this interval. This proves the theorem.

Now write

$$P(x) = \frac{1}{\pi}|x| - \tfrac{1}{2}\sin x,$$

$$Q(x) = \frac{1}{\pi}|x| + \tfrac{1}{2}\sin x.$$

We have

$$P(-\tfrac{1}{2}\pi) = 1, \qquad P(0) = P(\tfrac{1}{2}\pi) = 0;$$

$$Q(\tfrac{1}{2}\pi) = 1, \qquad Q(0) = Q(-\tfrac{1}{2}\pi) = 0.$$

Also

$$\underline{\lambda} = \lim_{y \to -0} -\frac{Q(y)}{P(y)} = \frac{\pi-2}{\pi+2},$$

$$\bar{\lambda} = \lim_{y \to +0} -\frac{Q(y)}{P(y)} = \frac{\pi+2}{\pi-2}.$$

Hence, if we take any number μ such that

$$\frac{\pi-2}{\pi+2} \leqslant \mu \leqslant \frac{\pi+2}{\pi-2},$$

and write

$$\begin{aligned} R(x) &= Q(x)+\mu P(x) \\ &= \frac{1}{\pi}(1+\mu)|x|+\tfrac{1}{2}(1-\mu)\sin x, \end{aligned}$$

$R(x)$ will take the value unity at the point $\frac{1}{2}\pi$ and have a double zero at the origin.

FREE STREAM-LINE FLOW PAST A VORTEX

By N. SIMMONS (*London*)

[Received 16 February 1939]

1. Introduction

THE determination of the influence of boundaries on the field of motion around a vortex is a problem which assumes an important aspect in wind-tunnel theory. This aspect has interested a number of writers*, and solutions have been obtained for both two-dimensional and three-dimensional fields. The methods used are, however, open to certain objections. Of these the chief relates to the use of image systems. Where the boundaries are fixed walls, on which the vanishing of the normal component of velocity is demanded, the problem may be resolved by this means. Where, however, the boundaries consist of free stream-lines, on which the condition imposed is the constancy of the pressure, the image method, strictly speaking, is no longer justifiable. Nevertheless, previous writers have used image systems for this case (of a type different from that for fixed boundaries), taking them to be permissible when the squares of the 'added velocities' may be neglected, and the deviations of the boundary from a straight line are small.† If the boundaries consist of both fixed walls and free stream-lines, as in the case of a jet, or a wind tunnel with open working section, the justification is less obvious.

In the present paper I solve the problem of two-dimensional flow round a vortex, the fluid issuing from between two parallel walls, making no assumptions beyond those usual in free stream-line theory. The problem of a stream bounded by free stream-lines only, which can be deduced as a limiting case from that of the jet, is also treated directly. The method used is not entirely new, but is similar to that employed in problems of free stream-line flow with singularities by Hopkinson (**2**), which paper seems to have attracted little attention.

It is proposed in a subsequent communication to deal with important particular cases, to show the application to wind-tunnel theory, and to compare the results with those of L. Prandtl. Also deferred to a later stage is the discussion of the case, approximating more closely to a wind tunnel of open working-section, where the

* See Durand (**1**), 2, 238 for a detailed list.

† Durand (**1**), 2, 241.

stream issues from the orifice formed by two parallel walls and returns to a collecting channel of similar form.

2. Stream issuing from between two parallel walls: preliminaries

Fluid issues as a two-dimensional jet from between two parallel walls CB, $C'D$ (Fig. 1), the free stream-lines being BA, DA': the points C, C', A, A' are all at infinity. There is a vortex in the jet, as shown.

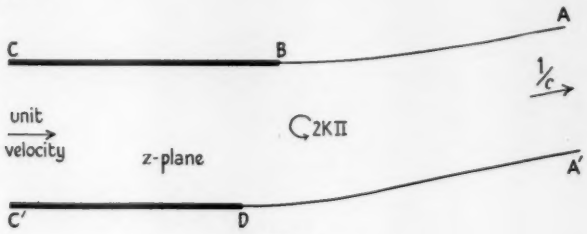

FIG. 1

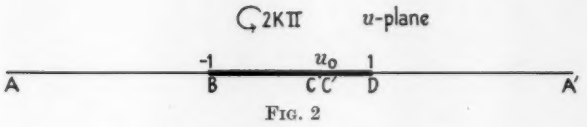

FIG. 2

For convenience, we take the distance between the walls as π units, and the initial velocity of the stream as unit velocity. This is tantamount to choosing special units of length and time, and so causes no real loss of generality. Let the coefficient of contraction of the jet be c: then the free stream-line velocity is $1/c$. Let the strength of the vortex be $2\kappa\pi$.

Take for origin O the mid-point of BD, with Ox in the direction of the undisturbed stream, and Oy perpendicular. Let z denote the complex variable in this plane. B, D are the points $\pm(d+\frac{1}{2}i\pi)$, where $2d$ is the overlap of the two walls.

Let Q denote the magnitude of the fluid-velocity at any point, and α its inclination to Ox. Let w represent the complex potential, on that convention in which

$$Qe^{-i\alpha} = \frac{dw}{dz}. \tag{2.1}$$

We seek a conformal transformation $u = u(z)$ which shall represent the area of motion in the z-plane on an area in the u-plane in such a way that the boundary $ABCC'DA'$ may transform into the real axis. This transformation can always be chosen so that the fixed boundaries correspond to the segment of the real axis joining ± 1, while the point at infinity to which the stream proceeds (A, A') corresponds to $u = \infty$. The free stream-lines transform to the portions of the real axis of u outside the segment $(-1, 1)$. Let (C, C'), the point at infinity whence the stream comes, correspond to u_0 (Fig. 2).

The singularity in w at the point z_1 representing the vortex will transform into a similar singularity in w at u_1, the point into which z_1 transforms. Motion in the u-plane represented by the complex potential w is therefore due to a vortex at u_1 of strength $2\kappa\pi$ and a source at u_0 discharging fluid at rate π, while the real axis is a bounding stream-line. Taking account of the image system we obtain

$$w = \log(u-u_0) - i\kappa \log(u-u_1) + i\kappa \log(u-\bar{u}_1) + \text{constant}, \quad (2.2)$$

and

$$\frac{dw}{du} = \frac{1}{u-u_0} - \frac{i\kappa(u_1-\bar{u}_1)}{(u-u_1)(u-\bar{u}_1)} = \frac{\text{quadratic}}{(u-u_0)(u-u_1)(u-\bar{u}_1)}. \quad (2.3)$$

The quadratic numerator has real coefficients, and so its zeros are either conjugate complex numbers, or both real: we shall say that

FIG. 3

the motion is of the first or second type accordingly. A zero corresponds to a stagnation point if it falls in the upper half-plane. Thus, in motions of the first type, there is one stagnation point which occurs at an internal point of the fluid (Fig. 3). In motions of the second type, there are two stagnation points, which occur on the boundary of the area of motion (Fig. 4). These must clearly be

situated on the fixed boundaries, and, considering the changes in the direction of flow along the walls, we easily find that both stagnation points must lie on the same wall. This is the upper or the lower according as $\kappa \gtrless 0$, the zeros of the quadratic lying, accordingly, in the segment $(-1, u_0)$ or $(u_0, 1)$. Figs. 3 and 4 are drawn for the case $\kappa > 0$.

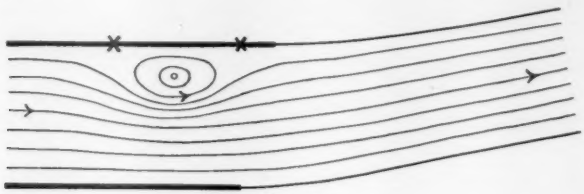

FIG. 4

The first type is of the greater interest and will be developed fully. The method of dealing with the second type is similar and presents nothing novel. The condition for the motion to be of the first type is found to reduce to

$$(1-\kappa^2)\Im\{u_1\}+2\kappa[\Re\{u_1\}-u_0] > 0. \tag{2.4}$$

3. Development of the method

Define a function Ω by the relation

$$\Omega = -\log\frac{dw}{dz}. \tag{3.1}$$

Then, from (2.1), $$\Omega = -\log Q+i\alpha. \tag{3.2}$$

On the real axis, between ± 1, Ω must be real, while, outside this segment, the real part of Ω must be constant. In the neighbourhood of $u = \pm 1$, Ω must be expansible in the form $(\pm 1-u)^{\frac{1}{2}} \times$ regular function. At infinity Ω must be finite. If u_1' denote the stagnation point, Ω must have logarithmic singularities at u_1, u_1', with coefficients of the logarithmic parts ± 1 respectively, so that the relation between z and u may be conformal at these points.* Hence the function

$$\sqrt{(1-u^2)}\frac{d\Omega}{du}-\left[\frac{\sqrt{(1-u_1^2)}}{u-u_1}+\frac{\sqrt{(1-\bar{u}_1^2)}}{u-\bar{u}_1}-\frac{\sqrt{(1-u_1'^2)}}{u-u_1'}-\frac{\sqrt{(1-\bar{u}_1'^2)}}{u-\bar{u}_1'}\right]$$

is regular at every point of the upper half-plane, and real and continuous on the real axis. It can therefore, by the Riemann-Schwarz

* Hopkinson (**2**), 147.

principle of reflection, be analytically continued to every point of the lower half-plane. The function is thus seen to be regular at every point, including infinity, and so is a constant: its value is equal to that at infinity, viz. zero. Hence

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{(1-u^2)}}\left[\frac{\sqrt{(1-u_1^2)}}{u-u_1} + \frac{\sqrt{(1-\bar{u}_1^2)}}{u-\bar{u}_1} - \frac{\sqrt{(1-u_1'^2)}}{u-u_1'} - \frac{\sqrt{(1-\bar{u}_1'^2)}}{u-\bar{u}_1'}\right]. \tag{3.3}$$

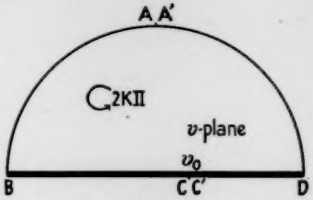

FIG. 5

It will be found simpler to transform the u-plane into a v-plane by means of the relation

$$u = \frac{2v}{1+v^2}. \tag{3.4}$$

The area of motion is then represented conformally on the upper half of the unit circle in the v-plane (Fig. 5). We find

$$\Omega = \log\left[\frac{v-v_1}{1-v_1 v}\,\frac{v-\bar{v}_1}{1-\bar{v}_1 v}\,\frac{1-v_1' v}{v-v_1'}\,\frac{1-\bar{v}_1' v}{v-\bar{v}_1'}\right] + \log c, \tag{3.5}$$

the constant of integration being determined by reference to point B or D. Hence

$$\frac{dw}{dz} = \frac{1}{c}\,\frac{(1-v_1 v)(1-\bar{v}_1 v)}{(v-v_1)(v-\bar{v}_1)}\,\frac{(v-v_1')(v-\bar{v}_1')}{(1-v_1' v)(1-\bar{v}_1' v)}. \tag{3.6}$$

Also, if v_0 denote the position of C in the v-plane, (2.3) gives

$$\frac{dw}{dv} = \frac{b(1+v_0^2)}{c(v-v_0)(1-v_0 v)}\,\frac{1-v^2}{1+v^2}\,\frac{(v-v_1')(1-v_1' v)(v-\bar{v}_1')(1-\bar{v}_1' v)}{(v-v_1)(1-v_1 v)(v-\bar{v}_1)(1-\bar{v}_1 v)}, \tag{3.7}$$

where

$$b = c\,\frac{(1+v_1^2)(1+\bar{v}_1^2)}{(1+v_1'^2)(1+\bar{v}_1'^2)}. \tag{3.8}$$

Define new constants, all real, as follows:

$$\begin{aligned} p &= |v_1|^2, \qquad q = -\Re\{v_1\}, \qquad r = \Im\{v_1\}, \\ p' &= |v_1'|^2, \qquad q' = -\Re\{v_1'\}. \end{aligned} \tag{3.9}$$

Then
$$\frac{dw}{dz} = \frac{(1+2qv+pv^2)(v^2+2q'v+p')}{c(v^2+2qv+p)(1+2q'v+p'v^2)}, \tag{3.10}$$

while, from (3.7) and (3.10),

$$\frac{dz}{dv} = \frac{b(1+v_0^2)}{(v-v_0)(1-v_0 v)} \frac{1-v^2}{1+v^2} \frac{(1+2q'v+p'v^2)^2}{(1+2qv+pv^2)^2}, \tag{3.11}$$

and from (3.8)
$$b = c\frac{(1-p)^2+4q^2}{(1-p')^2+4q'^2}. \tag{3.12}$$

The constants p, q, p', q' are not all independent, but are connected through (2.3). The relations between them can best be expressed by the identity

$$(v^2+2qv+p)(1+2qv+pv^2)+\frac{2h}{1+v_0^2}(v-v_0)(1-v_0 v)(1+v^2)$$
$$\equiv \frac{b}{c}(v^2+2q'v+p')(1+2q'v+p'v^2), \tag{3.13}$$

where
$$h = \kappa r(1-p). \tag{3.14}$$

In the following investigation, the four basic arbitrary constants determining the motion may most conveniently be taken to be p, q, v_0, κ. The values of these being assigned, u_0 follows, and it is necessary next to find p' and q'. The identity (3.13) yields the following quadratic equation for p':

$$\frac{1+p'^2}{p'} =$$
$$\frac{(1-p)^2+4q^2}{2(p-hu_0)} + \frac{[\{(1+p+2q)^2+4h(1-u_0)\}\{(1+p-2q)^2-4h(1+u_0)\}]^{\frac{1}{2}}}{2(p-hu_0)}. \tag{3.15}$$

The root which is numerically less than unity must be taken (see Fig. 5). Then

$$q' = \frac{p'}{1+p'}\frac{q(1+p)+h}{p-hu_0}. \tag{3.16}$$

When $v = i$, $\frac{dw}{dz} = \frac{1}{c}e^{-i\delta}$, where δ denotes the ultimate inclination of the stream to Ox. Hence

$$e^{i\delta} = \frac{(1-p-2iq)(1-p'+2iq')}{(1-p+2iq)(1-p'-2iq')}, \qquad \tan\tfrac{1}{2}\delta = \frac{2[(1-p)q'-(1-p')q]}{(1-p)(1-p')+4qq'}. \tag{3.17}$$

Now use of the identity (3.13) in (3.11) gives

$$\frac{dz}{dv} = \frac{c(1+v_0^2)(1-v^2)(1+2q'v+p'v^2)(v^2+2qv+p)}{(v-v_0)(1-v_0v)(1+v^2)(1+2qv+pv^2)(v^2+2q'v+p')} + \\ + \frac{2hc(1-v^2)(1+2q'v+p'v^2)}{(1+2qv+pv^2)^2(v^2+2q'v+p')}. \quad (3.18)$$

This can be put into the partial fractions

$$\frac{dz}{dv} = \frac{A_1}{v-v_0} - \frac{A_2v_0}{1-v_0v} + \frac{A_3v+A_4}{1+v^2} + \\ + 2hc\left[\frac{C_1v+C_2}{1+2qv+pv^2} + \frac{A_5v+A_6}{v^2+2q'v+p'} + \frac{(1-v^2)(1+2q'v+p'v^2)}{(1+2qv+pv^2)^2(v^2+2q'v+p')}\right], \quad (3.19)$$

where all the new constants are real, and

$$A_1 = b\left(\frac{1+2q'v_0+p'v_0^2}{1+2qv_0+pv_0^2}\right)^2, \qquad A_2 = b\left(\frac{v_0^2+2q'v_0+p'}{v_0^2+2qv_0+p}\right)^2, \quad (3.20)$$

$$-A_3+iA_4 = 2b\left(\frac{1-p'+2iq'}{1-p+2iq}\right)^2,$$

$$\frac{C_1}{v_1}+C_2 = -\frac{c}{b(1-p)}\left(\frac{1+2qv_1+pv_1^2}{1+2q'v_1+p'v_1^2}\right)^2, \quad (3.21)$$

determining C_1 and C_2.

Making use of the fact that dz/dv has no singularity at either zero of $v^2+2q'v+p'$, we find that (3.19) reduces to

$$\frac{dz}{dv} = \frac{A_1}{v-v_0} - \frac{A_2v_0}{1-v_0v} + \frac{A_3v+A_4}{1+v^2} + \\ + 2hc\left[\frac{(C_1+C_3)v+(C_2+C_4)}{1+2qv+pv^2} + \frac{C_5v+C_6}{(1+2qv+pv^2)^2}\right], \quad (3.22)$$

where $$\frac{C_5}{v_1}+C_6 = \frac{(v_1^2-1)(v_1^2+2q'v_1+p')}{v_1^2(1+2q'v_1+p'v_1^2)},$$

and

$$(1+2q'v_1+p'v_1^2)\left[\left(\frac{C_3}{v_1}+C_4\right)(p+qv_1)+\tfrac{1}{2}C_5v_1\right] \\ = (v_1^2-1)(p'+q'v_1)-(v_1^2+2q'v_1+p')-v_1(1+q'v_1)(C_5+C_6v_1),$$

determining C_3, C_4, C_5, C_6.

Defining further real constants by the relations

$$C_7 = \frac{C_6 p - C_5 q}{2r^2}, \qquad C_8 = \frac{C_6 q - C_5}{2r^2},$$

$$C_9 = \frac{1}{2r}\left[C_2 + C_4 + C_7 - (C_1 + C_3)\frac{q}{p}\right], \qquad C_{10} = \frac{C_1 + C_3}{2p}, \quad (3.23)$$

we obtain

$$\frac{dz}{dv} = \frac{A_1}{v - v_0} - \frac{A_2 v_0}{1 - v_0 v} + \frac{A_3 v + A_4}{1 + v^2} + \\ + 2hc\left[\frac{(C_1 + C_3)(v + q/p) + 2rC_9}{1 + 2qv + pv^2} + \frac{d}{dv}\left(\frac{C_7 v + C_8}{1 + 2qv + pv^2}\right)\right].$$

Integrating this, we have

z+constant

$$= A_1 \log(v - v_0) + A_2 \log(1 - v_0 v) + \tfrac{1}{2}A_3 \log(1 + v^2) + A_4 \tan^{-1} v + \\ + 2hc\left[\frac{C_7 v + C_8}{1 + 2qv + pv^2} + 2C_9 \tan^{-1}\frac{pv + q}{r} + C_{10} \log(1 + 2qv + pv^2)\right].$$

The functions $\tan^{-1} v$, $\tan^{-1}\{(pv+q)/r\}$ are uniform within the semicircular area (Fig. 5). Take the branches, the values of which at the origin are, respectively, 0 and (pr. arg $v_1 - \frac{1}{2}\pi$). The functions may then be written in the alternative forms

$$\tfrac{1}{2}i \log\frac{1 - iv}{1 + iv} \quad \text{and} \quad \tfrac{1}{2}i \log\frac{r - i(pv + q)}{r + i(pv + q)} \qquad \text{(principal values)}$$

respectively. Take the principal branches of the other logarithms also.

When $v = \pm 1$, $z = \mp(d + \frac{1}{2}i\pi)$. Subtracting the equations resulting from the substitution of these values, we find $A_1 = 1$. Hence

$$b = \left(\frac{1 + 2qv_0 + pv_0^2}{1 + 2q'v_0 + p'v_0^2}\right)^2,$$

$$c = \frac{(v_0^2 + 2q'v_0 + p')(1 + 2qv_0 + pv_0^2)}{(v_0^2 + 2qv_0 + p)(1 + 2q'v_0 + p'v_0^2)}, \quad \text{and} \quad A_2 = c^2. \quad (3.24)$$

Also, by making use of a relation derivable from the identity (3.13),

$$-A_3 + iA_4 = 2ce^{i\delta}. \quad (3.25)$$

Thus the relation between z and v is

$$z+a+\tfrac{1}{2}i\pi = \log(v-v_0)+c^2\log(1-v_0 v)-c\cos\delta\log\frac{1+v^2}{2}+2c\sin\delta\tan^{-1}v+ +2hc\left[\frac{C_7 v+C_8}{1+2qv+pv^2}+2C_9\tan^{-1}\frac{pv+q}{r}+C_{10}\log(1+2qv+pv^2)\right]. \quad (3.26)$$

where, by adding the equations for $v = \pm 1$,

$$a = \frac{1+c^2}{2}\log(1-v_0^2)+ +2hc\left[\frac{C_8(1+p)-C_7 2q}{(1+p)^2-4q^2}+C_9\left\{\tan^{-1}\frac{p+q}{r}-\tan^{-1}\frac{p-q}{r}\right\}+ +\tfrac{1}{2}C_{10}\log\{(1+p)^2-4q^2\}\right], \quad (3.27)$$

and, by subtracting the equations,

$$d = \frac{1+c^2}{2}\log\frac{1+v_0}{1-v_0}-\tfrac{1}{2}\pi c\sin\delta+ +2hc\left[\frac{C_8 2q-C_7(1+p)}{(1+p)^2-4q^2}-C_9\left\{\tan^{-1}\frac{p+q}{r}+\tan^{-1}\frac{p-q}{r}\right\}+ +\tfrac{1}{2}C_{10}\log\frac{1+p-2q}{1+p+2q}\right]. \quad (3.28)$$

4. The free stream-lines and their asymptotes

The free stream-lines in the z-plane correspond to the curved boundary of the semicircle in the v-plane (Fig. 5), on which $v = e^{i\theta}$, the lower one corresponding to the part on which $0 \leqslant \theta < \frac{1}{2}\pi$ and the upper one to the part on which $\frac{1}{2}\pi < \theta \leqslant \pi$.

Defining two real functions $f_1(\theta)$, $f_2(\theta)$, continuous on the semicircular arc, by means of the relation

$$f_1(\theta)+if_2(\theta) = \log(e^{i\theta}-v_0)+c^2\log(1-v_0 e^{i\theta})-c\cos\delta\log\frac{e^{i\theta}+i}{2}+ +ic\sin\delta\log(e^{i\theta}+i)+2hc\left[\frac{C_7 e^{i\theta}+C_8}{1+2qe^{i\theta}+pe^{2i\theta}}+ +iC_9\log\frac{r-i(pe^{i\theta}+q)}{r+i(pe^{i\theta}+q)}+C_{10}\log(1+2qe^{i\theta}+pe^{2i\theta})\right], \quad (4.1)$$

we deduce the equations of the lower and upper free stream-lines in the parametric form

$$x+a+c\cos\delta\log\{2\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)\}-c(\tfrac{3}{4}\pi+\tfrac{1}{2}\theta)\sin\delta = f_1(\theta),$$

$$y+\tfrac{1}{2}\pi-c(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)\cos\delta+c\sin\delta\log\{2\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)\} = f_2(\theta), \quad (4.2)$$

and

$$x+a+c\cos\delta\log\{2\sin(\tfrac{1}{2}\theta-\tfrac{1}{4}\pi)\}-c(\tfrac{1}{2}\theta-\tfrac{1}{4}\pi)\sin\delta = f_1(\theta),$$

$$y+\tfrac{1}{2}\pi+c(\tfrac{1}{2}\theta-\tfrac{5}{4}\pi)\cos\delta+c\sin\delta\log\{2\sin(\tfrac{1}{2}\theta-\tfrac{1}{4}\pi)\} = f_2(\theta),$$

respectively. The equations of the lower and upper asymptotes are respectively

$$\{x+a-f_1(\tfrac{1}{2}\pi)\}\sec\delta-\pi c\tan\delta = \{y+\tfrac{1}{2}\pi-f_2(\tfrac{1}{2}\pi)\}\operatorname{cosec}\delta,$$

$$\{x+a-f_1(\tfrac{1}{2}\pi)\}\sec\delta+\pi c\cot\delta = \{y+\tfrac{1}{2}\pi-f_2(\tfrac{1}{2}\pi)\}\operatorname{cosec}\delta, \quad (4.3)$$

where

$$f_1(\tfrac{1}{2}\pi)+if_2(\tfrac{1}{2}\pi)$$

$$= \log(i-v_0)+c^2\log(1-v_0 i)-\tfrac{1}{2}\pi ci\cos\delta-\tfrac{1}{2}\pi c\sin\delta+ci\sin\delta\log 2+$$

$$+2hc\left[\frac{C_7 i+C_8}{1-p+2iq}+iC_9\log\frac{r+p-iq}{r-p+iq}+C_{10}\log(1-p+2iq)\right].$$

Putting the constant in equation (2.2) equal to $-\frac{1}{2}i\pi$ we have

$$w = \log i(u_0-u)-i\kappa[\log(|u-u_1|/|u-\bar{u}_1|)+i\arg(u-u_1)-i\arg(u-\bar{u}_1)]. \quad (4.4)$$

The second argument is one-valued in the area of motion and so can always be given its principal value. The first, however, is many-valued.

On the free stream-lines, $v = e^{i\theta}$ and $u = \sec\theta$, so that

$$w = \log i(u_0-\sec\theta)+\kappa[\arg(\sec\theta-u_1)-\arg(\sec\theta-\bar{u}_1)].$$

The lower free stream-line corresponds to $\Im\{w\} = -\frac{1}{2}\pi$ and the upper to $\Im\{w\} = \frac{1}{2}\pi$. On both

$$\Re\{w\} = \log|\sec\theta-u_0|+\kappa[\arg(\sec\theta-u_1)-\arg(\sec\theta-\bar{u}_1)].$$

If s denote distance along the stream-line measured from D or B,

$$s = c\log\left|\frac{\sec\theta-u_0}{\pm 1-u_0}\right|+\kappa c\left[\arg\left(\frac{\sec\theta-u_1}{\pm 1-u_1}\right)-\arg\left(\frac{\sec\theta-\bar{u}_1}{\pm 1-\bar{u}_1}\right)\right]. \quad (4.5)$$

Also, if ψ denote the angle of contingence at a point of the stream-line, then

$$ce^{i\psi} = \left(\frac{dz}{dw}\right)_{v=e^{i\theta}},$$

whence
$$e^{i\psi} = \frac{(e^{2i\theta}+2qe^{i\theta}+p)(1+2q'e^{i\theta}+p'e^{2i\theta})}{(1+2qe^{i\theta}+pe^{2i\theta})(e^{2i\theta}+2q'e^{i\theta}+p')}. \tag{4.6}$$

The intrinsic equations of the free stream-lines are thus obtained in terms of the parameter θ.

5. Determination of constants associated with the vortex

The position of the vortex being denoted by z_1, we have, putting $z = z_1$, $v = v_1 = -q+ir$ in (3.26),

$$z_1+a+\tfrac{1}{2}i\pi =$$
$$\log(-v_0-q+ir)+c^2\log(1+v_0q-iv_0r)-c\cos\delta\log\frac{1+q^2-r^2-2iqr}{2}+$$
$$+ic\sin\delta\log\frac{1+r+iq}{1-r-iq}+2hc\left[\frac{-C_7q+C_8+iC_7r}{(1-p)(1+p-2q^2+2iqr)}+\right.$$
$$\left.+iC_9\log\frac{r(1+p)-iq(1-p)}{r(1-p)+iq(1-p)}+C_{10}\log\{(1-p)(1+p-2q^2+2iqr)\}\right].$$

Hence

$$x_1+a =$$
$$\tfrac{1}{2}\log(v_0^2+2v_0q+p)+\tfrac{1}{2}c^2\log(1+2v_0q+v_0^2p)-\tfrac{1}{2}c\cos\delta\log\frac{(1-p)^2+4q^2}{4}-$$
$$-c\sin\delta\arg(1-p+2iq)+2hc\left[\frac{C_8(1+p-2q^2)-C_7(1-p)q}{(1-p)\{(1-p)^2+4r^2\}}-\right.$$
$$\left.-C_9\arg\{2r^2-p(1-p)-2iqr\}+C_{10}\log[(1-p)\sqrt{\{(1-p)^2+4r^2\}}]\right],$$

and

$$y_1+\tfrac{1}{2}\pi = \arg(-v_0-q+ir)+c^2\arg(1+v_0q-iv_0r)-$$
$$-c\cos\delta\arg(1+p-2r^2-2iqr)+c\sin\delta\log\frac{1+p+2r}{\sqrt{\{(1-p)^2+4q^2\}}}+$$
$$+2hc\left[\frac{\{C_7(1+p)-C_8\,2q\}r}{(1-p)\{(1-p)^2+4r^2\}}+C_9\log\frac{\sqrt{\{(1-p)^2+4r^2\}}}{1-p}+\right.$$
$$\left.+C_{10}\arg(1+p-2q^2+2iqr)\right]. \tag{5.1}$$

Principal arguments must be taken in all cases.

Now let U_1, V_1 denote the components of the velocity of the vortex, if free, or of the fluid at the vortex, if bound. If we imagine the (bound) vortex replaced by a small two-dimensional body, e.g. an aerofoil, round which the circulation is $2\kappa\pi$, there will be a force on the body, say of magnitude F_1 at angle β_1 to Ox_1, given by the Kutta-Joukowski law

$$F_1 e^{i\beta_1} = -i2\kappa\pi Q_1 e^{i\alpha_1} = 2\kappa\pi(V_1 - iU_1) \tag{5.2}$$

(taking the unit of density to be that of the fluid). Further,

$$\begin{aligned} F_1 e^{-i\beta_1} &= \frac{i}{2}\int_{(z_1)} \left(\frac{dw}{dz}\right)^2 dz \\ &= \frac{i}{2}\int_{(v_1)} \frac{b}{c^2}\,\frac{1+v_0^2}{(v-v_0)(1-v_0 v)}\,\frac{1-v^2}{1+v^2}\left(\frac{v^2+2q'v+p'}{v^2+2qv+p}\right)^2 dv \\ &= \frac{i}{2c^2}\int_{(1/v_1)} \frac{b(1+v_0^2)v^2}{(1-v_0 v)(v-v_0)}\,\frac{v^2-1}{v^2+1}\left(\frac{1+2q'v+p'v^2}{1+2qv+pv^2}\right)^2\left(-\frac{1}{v^2}\right) dv \\ &= \frac{i}{2c^2}\int_{(1/v_1)} \frac{dz}{dv}\,dv = -\frac{2\pi h}{c}(iC_9 + C_{10}). \end{aligned}$$

Hence
$$F_1 e^{i\beta_1} = \frac{2\pi h}{c}(-C_{10} + iC_9) \tag{5.3}$$

and
$$U_1 + iV_1 = Q_1 e^{i\alpha_1} = -\frac{r(1-p)}{c}(C_9 + iC_{10}). \tag{5.4}$$

We shall determine β_1 so that $0 \leqslant \beta_1 \leqslant \pi$, and allow F_1 to take both positive and negative values.

Again, if w_∞ denote the potential due to the vortex in infinite fluid, and w_c the complementary potential $w - w_\infty$, the radius of curvature R_1 of the stream at the vortex is given by

$$\frac{1}{R_1} = -\frac{1}{Q_1}\,\Im\left\{e^{2i\alpha}\frac{d^2 w_c}{dz^2}\right\}_1.$$

We find that

$$w_c = \log(u-u_0) + i\kappa\log\left\{\sum_{n=1}^{\infty}\frac{c_n}{n!}(u-u_1)^{n-1}\right\} + i\kappa\log(u-\bar{u}_1) - \tfrac{1}{2}i\pi,$$

where
$$c_n = \left(\frac{d^n z}{du^n}\right)_1 \quad (n = 1, 2, \ldots). \tag{5.5}$$

This leads to the result

$$\frac{1}{R_1}=\frac{1}{Q_1}\Im\left\{e^{2i\alpha_1}\left[\frac{2(1-p)(1+v_1^2)^2(\Delta q+v_1\Delta p+v_1^2 q\Delta p-v_1^2 p\Delta q)}{bc_1(1+2q'v_1+p'v_1^2)^3}+\right.\right.$$

$$\left.\left.+\frac{i\kappa}{c_1^4}\left(\tfrac{3}{4}c_2^2-\frac{c_1c_3}{3}\right)+\frac{i\kappa}{c_1^2(u_1-\bar{u}_1)}\left(\frac{c_2}{c_1}+\frac{1}{u_1-\bar{u}_1}\right)\right]\right\}, \quad (5.6)$$

where $\Delta p \equiv p'-p$ and $\Delta q \equiv q'-q$.

To apply these formulae in practice, it is necessary to deduce the constants p, q, v_0 from the physical data d, x_1, y_1, κ. The equations are tractable in three main cases:

(i) if the vortex lies appreciably downstream;
(ii) if the vortex lies near an edge of the orifice;
(iii) if the vortex is weak.

The solutions for these cases (with overlap zero) will be given later.

6. Problem of stream with free boundaries only

The solution for the case where the fluid is limited by free boundaries only will now be obtained for general values of the vortex strength.

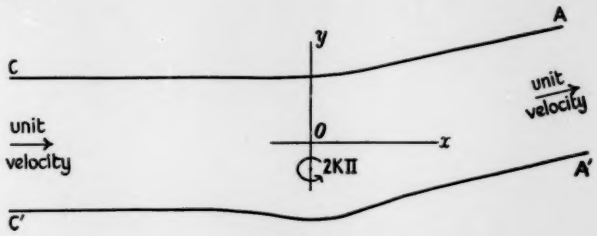

FIG. 6

The ultimate width of the stream is equal to the initial width, π units, say. Take the velocity along the free stream-lines as unit velocity. Let the strength of the vortex be $2\kappa\pi$. Let the angle through which the stream is turned be δ.

Take the mid-line of the undisturbed stream as x-axis. Let the y-axis be chosen to pass through the vortex, whose coordinates are $(0, y_1)$. Let w represent the complex potential.

We seek a conformal transformation $u = u(z)$ which shall represent the area of motion on the upper u half-plane so that C, C' transform to the origin, A, A' to the point at infinity, while $|u_1| = 1$.

Motion in the u-plane represented by the complex potential w is due to a vortex at u_1 of strength $2\kappa\pi$ and a source at the origin discharging fluid at rate π, while the real axis is a bounding streamline. Taking account of the image system, we obtain

$$w = \log u - i\kappa \log(u-u_1) + i\kappa \log(u-\bar{u}_1) + \text{constant} \tag{6.1}$$

and

$$\frac{dw}{du} = \frac{1}{u} - \frac{i\kappa(u_1-\bar{u}_1)}{(u-u_1)(u-\bar{u}_1)} = \frac{(u-u_1')(u-\bar{u}_1')}{u(u-u_1)(u-\bar{u}_1)}$$

where u_1' represents the stagnation point

We deduce that

$$u_1'+\bar{u}_1' = u_1+\bar{u}_1+i\kappa(u_1-\bar{u}_1) \quad \text{and} \quad |u_1'| = 1. \tag{6.2}$$

Now Ω must be purely imaginary on the real axis of u, be finite at infinity, and must have logarithmic singularities at u_1, u_1', with coefficients of the logarithmic parts ± 1 respectively. By an argument similar to that used to establish (3.3) we find

$$\frac{d\Omega}{du} = \frac{1}{u-u_1} - \frac{1}{u-\bar{u}_1} - \frac{1}{u-u_1'} + \frac{1}{u-\bar{u}_1'}. \tag{6.3}$$

Integrating, we obtain

$$\frac{dz}{dw} = e^{i\delta}\frac{(u-u_1)(u-\bar{u}_1')}{(u-\bar{u}_1)(u-u_1')}. \tag{6.4}$$

Putting $u = 0$ in this we find

$$\frac{u_1'}{\bar{u}_1'} = e^{i\delta}\frac{u_1}{\bar{u}_1},$$

so that we may write

$$u_1 = e^{i(\lambda-\frac{1}{2}\delta)}, \qquad u_1' = e^{i(\lambda+\frac{1}{2}\delta)}, \tag{6.5}$$

where λ is real. Equation (6.2) now gives

$$\kappa = \frac{2\sin\lambda\sin\frac{1}{2}\delta}{\sin(\lambda-\frac{1}{2}\delta)}. \tag{6.6}$$

From (6.1) and (6.4) it follows that

$$\frac{dz}{du} = \frac{e^{i\delta}}{u}\left(\frac{u-\bar{u}_1'}{u-\bar{u}_1}\right)^2.$$

Integrating this equation, we obtain the required relation between z and u, namely

$$z+a_1+ia_2 = \log u + (e^{i\delta}-1)\log(u-\bar{u}_1) - (e^{i\delta/2}-1)^2\bar{u}_1/(u-\bar{u}_1), \tag{6.7}$$

where the logarithms take their principal values, and a_1, a_2 are real constants given by

$$a_1 = -2\sin^2\tfrac{1}{2}\delta\log\{2\sin(\lambda-\tfrac{1}{4}\delta)\}-\tfrac{1}{2}\pi\sin\delta+\frac{2\sin^2\tfrac{1}{4}\delta}{\sin(\lambda-\tfrac{1}{4}\delta)}\sin(\tfrac{3}{4}\delta-\lambda),$$

$$a_2 = \tfrac{1}{2}\pi-2\sin^2\tfrac{1}{2}\delta\{\pi-\lambda+\tfrac{1}{4}\delta+\tan\tfrac{1}{4}\delta\}. \tag{6.8}$$

We deduce that

$$y_1 = -(\tfrac{1}{2}\pi-\lambda+\tfrac{1}{4}\delta)\cos\delta+\sin\delta\log\{2\sin(\lambda-\tfrac{1}{4}\delta)\}-\frac{2\sin^2\tfrac{1}{4}\delta\cos(\lambda+\tfrac{1}{4}\delta)}{\sin(\lambda-\tfrac{1}{4}\delta)}. \tag{6.9}$$

To determine the quantities U_1, V_1, F_1, β_1, we have, as in § 5,

$$F_1 e^{i\beta_1} = -2i\kappa\pi Q_1 e^{i\alpha_1},$$

and

$$F_1 e^{-i\beta_1} = \frac{i}{2}\int_{(z_1)}\left(\frac{dw}{dz}\right)^2 dz. \tag{6.10}$$

These formulae lead to the results

$$F_1 e^{i\beta_1} = -2\pi\sin\tfrac{1}{2}\delta\, e^{i(\pi+\delta)/2} \tag{6.11}$$

and

$$U_1+iV_1 = Q_1 e^{i\alpha_1} = \frac{\sin(\lambda-\tfrac{1}{4}\delta)\cos\tfrac{1}{4}\delta}{\sin\lambda}e^{i\delta/2},$$

so that the direction of motion is equally inclined to the initial and final directions of the stream.

We again have the formula

$$\frac{1}{R_1} = -\frac{1}{Q_1}\Im\left\{e^{2i\alpha}\frac{d^2w_c}{dz^2}\right\}_1$$

and, after some reduction, obtain

$$\frac{1}{R_1} = \frac{\kappa\sin^3(\lambda-\tfrac{1}{4}\delta)}{3\sin^3\lambda\cos\tfrac{1}{4}\delta}\left[\operatorname{cosec}^2\lambda-\tfrac{1}{4}\cot^2(\lambda-\tfrac{1}{4}\delta)+\frac{3\kappa\cos\lambda}{4\sin^3\lambda}+\frac{\kappa}{16\sin^4\lambda}\right]. \tag{6.12}$$

From (6.1) and (6.4) it is seen that the equations of the free stream-lines are

$$s = \log|u|+\kappa\arg(u-u_1)-\kappa\arg(u-\bar{u}_1),$$

$$e^{i(\psi-\delta)} = \frac{(u-u_1)(u-\bar{u}_1')}{(u-\bar{u}_1)(u-u_1')}, \tag{6.13}$$

where u is a real parameter, positive values of u giving the lower stream-line, and negative values the upper.

It is easily shown that one of the free stream-lines is inflected

unless $\lambda = \frac{1}{2}\pi$. The inflected line is the lower or upper according as $\lambda < \frac{1}{2}\pi$ or $> \frac{1}{2}\pi$.

When the vortex is weak, i.e. κ is small, we have the first approximations

$$\delta = 2\kappa, \qquad \lambda = \tfrac{1}{2}\pi + y_1, \qquad u_1 = ie^{iy_1} = u_1',$$

$$a_1 = -\kappa\pi, \qquad a_2 = \tfrac{1}{2}\pi,$$

$$\Delta U \equiv U_1 - 1 = \tfrac{1}{2}\kappa \tan y_1, \qquad \Delta V \equiv V_1 = \kappa,$$

$$F_1 = -2\kappa\pi, \qquad \beta_1 = \tfrac{1}{2}\pi + \alpha_1,$$

$$\frac{1}{R_1} = \kappa(\tfrac{1}{3} + \tfrac{1}{4}\tan^2 y_1). \tag{6.14}$$

The equations for the free stream-lines reduce to

$$\psi = \kappa \frac{\sin y_1 \pm e^s}{\sin y_1 \pm \cosh s}. \tag{6.15}$$

7. Conclusion

In the above work the units have been chosen so as to give certain parameters convenient values. The results can, however, easily be generalized. If the distance between the fixed boundaries (or initial width of stream) be k, the general velocity U_0, and the fluid density ρ, we have merely to put $\pi x_1/k$ for x_1, $\pi y_1/k$ for y_1, $\kappa\pi/U_0 k$ for κ, U/U_0 for U, V/U_0 for V, $\pi F_1/\rho U_0^2 k$ for F_1, and so on.

The writer, in conclusion, wishes to express his indebtedness to Dr. R. C. Knight for supervising the above investigation and for valuable suggestions.

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FREE STREAM-LINE FLOW PAST VORTICES AND AEROFOILS

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[Received 16 February 1939]

1. Introduction

In a previous paper (4) I have discussed the problem of the influence of free boundaries on the field of flow around a vortex in two dimensions. The solution has an important application in aerodynamic work, where it is usual to replace an aerofoil by its equivalent vortex system, whether in two or three dimensions, and by investigating the motion of the fluid in its neighbourhood, to determine the necessary corrections to the lift and drag coefficients, due to the aerofoil being placed in a finite stream instead of in infinite fluid. We limit ourselves here to two-dimensional motion, the aerofoil being replaceable by a single rectilinear vortex.

The problem we investigate is concerned with an aerofoil or vortex in a stream of finite breadth issuing from between two parallel walls. In particular, as indicated in § 5 of (4) we can deal with three cases:

(a) (i) if the vortex lies appreciably downstream;
(ii) if the vortex lies near an edge of the orifice;
(iii) if the vortex is weak.

There is also the problem (b) of the aerofoil in the free stream, which has been shown to be soluble without restriction.* This is, of course, the limiting case of (a) (i) and I shall not discuss it in detail. The approximations in each case entail some considerable reduction, which, although different in the three, follows the same general lines. I shall therefore be content to deal in some detail with (a) (i) in § 2, merely indicating in §§ 3, 4 the type of approximation required for (a) (ii) and (a) (iii) respectively. The results for all three cases are generalized in § 5.

We have observed† that Prandtl's method is open to objection in its use of image systems. Also the solution thus obtained and discussed by Toussaint‡, applying primarily to problem (b) is used for the more practical problem (a), the assumption being implied that the fluid emerges from the walls in a direction parallel to them. It is reasonable to assume this if the distance of the aerofoil from

* (4) § 6. † (4) § 1. ‡ (2) 302–5.

the mouth of the tunnel is sufficiently large. In fact, the effect of the mouth is found to be negligible only if this distance exceeds 1·5 times the distance between the walls. Another assumption made in Toussaint's discussion we leave to § 7.

From the generalized results of § 5, we deduce formulae for the corrections to lift and drag coefficients of a two-dimensional aerofoil (*a*) in the mouth of a wind tunnel, and (*b*) in a free stream, without these assumptions. We compare them with the formulae given by Toussaint* for the symmetrical case, and suggest that his results require modification.

The notation employed is, throughout, that of (**4**).

2. When the vortex lies appreciably downstream

A first approximation will be obtained, but it will be necessary initially to find some constants to second approximations.

In the notation of (**4**), $|v_1-i|$ is small.

Put
$$v_1 = i(1-\epsilon e^{i\omega}). \tag{2.1}$$

Equations (3.15), (3.16) of (**4**) lead to

$$v_1' = i(1-\epsilon e^{-i\omega'}), \tag{2.2}$$

where
$$\cos\omega' = [(1-\kappa^2)\cos^2\omega+2\kappa\cos\omega\sin\omega]^{\frac{1}{2}}, \tag{2.3}$$
$$\sin\omega' = \kappa\cos\omega-\sin\omega.$$

We have, to first approximations,

$$1-p = 2\epsilon\cos\omega, \qquad q = -\epsilon\sin\omega, \qquad r = 1,$$
$$1-p' = 2\epsilon\cos\omega', \qquad q' = \epsilon\sin\omega', \qquad h = 2\kappa\epsilon\cos\omega. \tag{2.4}$$

Equation (3.17) of (**4**) yields

$$\delta = 2(\omega+\omega'). \tag{2.5}$$

Also, from (3.24) of (**4**),
$$b = c = 1. \tag{2.6}$$

We next evaluate the constants C_n, and deduce the following values of the physical constants:

$$\kappa = 2\sin\tfrac{1}{2}(\omega+\omega')\cos\tfrac{1}{2}(\omega-\omega')\sec\omega,$$
$$a = \log(1-v_0^2)-2\sin^2(\omega+\omega')\log 2+O(\epsilon),$$
$$d = \log\frac{1+v_0}{1-v_0}+O(\epsilon),$$

* (**2**) 304–5.

$$x_1+a = \log(1+v_0^2)-\log\epsilon+\omega\sin(2\omega+2\omega')+$$
$$+2\sin^2\tfrac{1}{2}(\omega+\omega')\frac{\cos\omega'}{\cos\omega}+O(\epsilon\log\epsilon),$$

$$y_1 = -\omega\cos(2\omega+2\omega')+\sin(2\omega+2\omega')\log(2\cos\omega)+$$
$$+2\sin^2\tfrac{1}{2}(\omega+\omega')\frac{\sin\omega'}{\cos\omega}+O(\epsilon),$$

$$U_1+iV_1 = \frac{\cos\omega\cos\tfrac{1}{2}(\omega+\omega')}{\cos\tfrac{1}{2}(\omega-\omega')}e^{\frac{1}{2}i\delta},$$

$$F_1 e^{i\beta_1} = -i2\kappa\pi\frac{\cos\tfrac{1}{2}(\omega+\omega')\cos\omega}{\cos\tfrac{1}{2}(\omega-\omega')}e^{\frac{1}{2}i\delta}. \qquad (2.7)$$

(Note that $\log\epsilon$ is not necessarily large.)

For non-overlapping ends, $d = 0$, and so $v_0 = O(\epsilon)$. Thus, substituting $\omega+\omega' = \frac{1}{2}\delta$ and putting $\omega-\omega' = \pi-2\lambda$, we obtain the first approximations

$$\kappa = \frac{2\sin\lambda\sin\tfrac{1}{4}\delta}{\sin(\lambda-\tfrac{1}{4}\delta)},$$

$$x_1 = \log\frac{1}{\epsilon}+(\tfrac{1}{2}\pi-\lambda+\tfrac{1}{4}\delta)\sin\delta+2\sin^2\tfrac{1}{4}\delta\frac{\sin(\lambda+\tfrac{1}{4}\delta)}{\sin(\lambda-\tfrac{1}{4}\delta)}+2\sin^2\tfrac{1}{2}\delta\log 2,$$

$$y_1 = -(\tfrac{1}{2}\pi-\lambda+\tfrac{1}{4}\delta)\cos\delta+\sin\delta\log\{2\sin(\lambda-\tfrac{1}{4}\delta)\}-$$
$$-2\sin^2\tfrac{1}{4}\delta\frac{\cos(\lambda+\tfrac{1}{4}\delta)}{\sin(\lambda-\tfrac{1}{4}\delta)},$$

$$Q_1 e^{i\alpha_1} = U_1+iV_1 = \frac{\sin(\lambda-\tfrac{1}{4}\delta)\cos\tfrac{1}{4}\delta}{\sin\lambda}e^{\frac{1}{2}i\delta},$$

$$Q_1 = \frac{\sin(\lambda-\tfrac{1}{4}\delta)\cos\tfrac{1}{4}\delta}{\sin\lambda}, \qquad \alpha_1 = \tfrac{1}{2}\delta,$$

$$F_1 = -2\kappa\pi Q_1 = -2\pi\sin\tfrac{1}{2}\delta, \qquad \beta_1 = \tfrac{1}{2}\pi+\tfrac{1}{2}\delta. \qquad (2.8)$$

Comparison with § 6 of (4) shows that these results are the same as those for a jet with only free boundaries (when x_1, however, has no meaning).

In practice x_1, y_1, and κ are the given quantities. Equation (2.8) (i) then gives an expression for λ in terms of $\tan\delta$, so that, on substituting for λ in (2.8) (iii) an equation for δ is obtained. This must be solved approximately: the other constants will then follow.

For the intrinsic equations of the free stream-lines, (4.5) of (4) gives the first approximation

$$s = \log|\sec\theta|,$$

so that, on the lower and upper stream-lines,

$$e^s = \pm\sec\theta, \tag{2.9}$$

respectively. Also from (4.6) of (4)

$$e^{i\psi} = \frac{[\cos\theta-\epsilon\sin\omega-\epsilon\cos\omega\, e^{-i\theta}][\cos\theta+\epsilon\sin\omega'-\epsilon\cos\omega' e^{i\theta}]}{[\cos\theta-\epsilon\sin\omega-\epsilon\cos\omega\, e^{i\theta}][\cos\theta+\epsilon\sin\omega'-\epsilon\cos\omega' e^{-i\theta}]}.$$

Near the orifice, where $\epsilon\sec\theta$ is small, this equation reduces to

$$\psi = -4\epsilon\tan\theta\cos\lambda\sin\tfrac{1}{4}\delta. \tag{2.10}$$

From (2.9) and (2.10) we deduce the intrinsic equations of the lower and upper free stream-lines as

$$\psi = \mp 4\epsilon(e^{2s}-1)^{\frac{1}{2}}\cos\lambda\sin\tfrac{1}{4}\delta, \tag{2.11}$$

respectively. In the neighbourhood of the orifice the jet is symmetrical.

For the asymptotes, we find

$$f_1(\tfrac{1}{2}\pi) = -\tfrac{1}{2}\pi\sin\delta, \qquad f_2(\tfrac{1}{2}\pi) = (1-\cos\delta)\tfrac{1}{2}\pi+\sin\delta\log 2,$$

so that the equations are

$$\begin{aligned} &\text{(lower asymptote)} \quad y = x\tan\delta-\tfrac{1}{2}\pi\sec\delta+(2\sin\delta-\tan\delta)\log 2,\\ &\text{(upper asymptote)} \quad y = x\tan\delta+\tfrac{1}{2}\pi\sec\delta+(2\sin\delta-\tan\delta)\log 2. \end{aligned} \tag{2.12}$$

When the vortex is weak we have the first approximations

$$\begin{aligned} &x_1 = \log\frac{1}{\epsilon}, && y_1 = \lambda-\tfrac{1}{2}\pi, && \kappa = \tfrac{1}{2}\delta,\\ &Q_1 = U_1 = 1, && V_1 = \kappa, && \alpha_1 = \kappa,\\ &F_1 = -2\kappa\pi, && \beta_1 = \tfrac{1}{2}\pi, && \frac{1}{R_1} = \kappa(\tfrac{1}{3}+\tfrac{1}{4}\tan^2 y_1). \end{aligned} \tag{2.13}$$

For an accuracy of 1 per cent., $x_1 \geqslant \log 100 = \frac{3}{2}\pi$ approximately, i.e. the vortex must be situated at a distance from the orifice at least 1·5 times the width of the jet.

3. When the vortex lies near one edge of the orifice

The vortex is now near to B or D. Choosing the former without loss of generality, we have the position v_1 such that $|v_1+1|$ is small. We may put

$$v_1 = -1+\epsilon e^{i\xi}, \tag{3.1}$$

with the resulting values

$$\begin{aligned} &p = 1-2\epsilon\cos\xi+\epsilon^2, && q = 1-\epsilon\cos\xi,\\ &r = \epsilon\sin\xi, && h = \kappa\epsilon^2\sin 2\xi. \end{aligned} \tag{3.2}$$

We find that $\kappa = O(\epsilon^2)$ and then proceed in much the same way as for the preceding case, retaining where necessary higher approximations than the first. Omitting the rather laborious reduction, we find that, if we define

$$\alpha = 1-\sec\xi\left[\tfrac{1}{2}\cos 2\xi+\frac{1}{2}\left(1-\frac{4\kappa}{\epsilon^2}\sin 2\xi\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$
$$\beta = \tan\xi, \tag{3.3}$$

the position, velocity, etc. of the vortex in the z-plane may be represented by

$$\frac{x_1}{\kappa} = \frac{2\beta}{\alpha(2-\alpha)\{(1-\alpha)^2+\beta^2\}}[\tfrac{1}{4}\pi^2\alpha^2+4\alpha+\tfrac{1}{2}\beta^2-1]+$$
$$+\frac{1}{\beta(2-\alpha)\{(1-\alpha)^2+\beta^2\}}\Big[-3\alpha\{(1-\alpha)^2-\beta^2\}+$$
$$+\tfrac{1}{2}\alpha(1-2\alpha+\alpha^2+\beta^2-2\alpha\beta^2)-4\alpha\beta^2\log\frac{16}{1+\beta^2}-$$
$$-2\{\alpha^3-2\alpha^2(1-\beta^2)+\alpha(1+\beta^2)-2\beta^2(1+\beta^2)\}\frac{\tan^{-1}\beta}{\beta}\Big],$$
$$= \Phi_1(\alpha,\beta),$$
$$\frac{\frac{1}{2}\pi-y_1}{\kappa} = \frac{2\beta^2(4\alpha-1)}{\alpha(2-\alpha)\{(1-\alpha)^2+\beta^2\}}+$$
$$+\frac{1}{(2-\alpha)\{(1-\alpha)^2+\beta^2\}}\Big[\alpha\{(1-\alpha)^2-\beta^2\}+8\alpha\beta\tan^{-1}\beta-$$
$$-2\{\alpha^3-2\alpha^2(1-\beta^2)+\alpha(1+\beta^2)-2\beta^2(1+\beta^2)\}\frac{\log(1+\beta^2)}{\beta^2}\Big],$$
$$= \Phi_2(\alpha,\beta),$$
$$U_1 = 1+\frac{\alpha}{\beta^2}\frac{1-2\alpha+\alpha^2-4\beta^2+6\alpha\beta^2-\alpha^2\beta^2-\beta^4}{(\alpha-2)\{(1-\alpha)^2+\beta^2\}} = \Phi_3(\alpha,\beta),$$
$$V_1 = \frac{4\alpha\beta}{(2-\alpha)\{(1-\alpha)^2+\beta^2\}} = \Phi_4(\alpha,\beta),$$
$$F_1\cos\beta_1 = 2\kappa\pi\Phi_4(\alpha,\beta),$$
$$F_1\sin\beta_1 = -2\kappa\pi\Phi_3(\alpha,\beta). \tag{3.4}$$

Also
$$\delta = [2\kappa\Phi_4(\alpha,\beta)]^{\frac{1}{2}},$$

and the equations of the lower and upper free stream-lines at finite distances from the points of departure are

$$\psi = \delta(\tanh\tfrac{1}{2}s)^{\frac{1}{2}}, \qquad \psi = \delta(\coth\tfrac{1}{2}s)^{\frac{1}{2}} \tag{3.5}$$

respectively.

In any practical case x_1, $\frac{1}{2}\pi - y_1$, κ are the given quantities: α, β may be treated as parameters, and a table of corresponding values of x_1, $\frac{1}{2}\pi - y_1$, U_1, V_1, etc., constructed; β must be positive, while α is cosignal with κ. Also the inequality $\alpha < 1$ is useful when κ is positive.

4. When the vortex is weak

In this case κ is small. The process of approximation requires the constants to (at most) the second order, and these are readily obtained. We take $v_0 = O(\kappa)$ and find that this includes the case where the boundaries are non-overlapping. The values of the physical quantities for this case, deduced from §§ 3–5 of (4) are found to be

$$\Delta U = U_1 - 1$$
$$= \frac{\kappa}{2\sigma}\left[\sinh x_1 \tan y_1 + \frac{1}{\sigma}\sin y_1 \cos y_1 - 2\sin y_1 \left(\frac{\sigma + \sinh x_1}{\sigma + \cosh x_1}\right)^{\frac{1}{2}}\right],$$

$$\Delta V = V_1 = \frac{\kappa \cosh x_1}{2\sigma^2}(\sigma + \sinh x_1), \tag{4.1}$$

where

$$\sigma = |\cosh z_1| = (\cosh^2 x_1 - \sin^2 y_1)^{\frac{1}{2}} = (\sinh^2 x_1 + \cos^2 y_1)^{\frac{1}{2}},$$

$$F_1 = -2\kappa\pi, \qquad \beta_1 = \tfrac{1}{2}\pi + \frac{\kappa \cosh x_1}{2\sigma^2}(\sigma + \sinh x_1),$$

$$\frac{1}{R_1} = \kappa\left[\frac{1}{3} - \frac{1}{8}\left(1 - \frac{2\sinh x_1 \sec^2 y_1}{\sigma} + \frac{\sinh^2 x_1 \cosh^2 x_1 - \sin^2 y_1 \cos^2 y_1}{\sigma^4}\right)\right],$$

$$\delta = \frac{\kappa}{\sigma}\sqrt{\{(\sigma + \sinh x_1)(\sigma + \cosh x_1)\}}.$$

The equations of the lower and upper free stream-lines become respectively

$$\psi = \frac{\frac{1}{2}\delta(1 - e^{-2x})^{\frac{1}{2}}}{\sigma + \cosh x_1} \frac{e^{x - x_1}(\sigma + \cosh x_1) \pm e^{x_1}\sin y_1}{\cosh(x - x_1) \pm \sin y_1}.$$

In the symmetrical case, i.e. when the vortex lies on the mid-line, the equations become

$$\Delta U = 0, \qquad \Delta V = \tfrac{1}{2}\kappa(1 + \tanh x_1), \tag{4.2}$$
$$F_1 = -2\kappa\pi, \qquad \beta_1 = \tfrac{1}{2}\pi + \tfrac{1}{2}\kappa(1 + \tanh x_1),$$
$$\frac{1}{R_1} = \kappa[\tfrac{1}{3} - \tfrac{1}{8}(1 - \tanh x_1)^2],$$
$$\delta = \kappa\sqrt{\{2(1 + \tanh x_1)\}}, \qquad \psi = \frac{\delta(1 - e^{-2x})^{\frac{1}{2}}}{1 + e^{2(x_1 - x)}}.$$

Here, δ, ψ, ΔV, R_1, F_1, and $\beta_1 - \frac{1}{2}\pi$ must be odd functions of κ, and so the corresponding results are actually true to a second approximation.

5. Summary of results with full dimensions

In the preceding sections the units have been chosen so as to give certain parameters convenient values. If the distance between the fixed boundaries (or initial width of stream) be k, the general velocity U_0, and the fluid density ρ, we may generalize the results by putting $\pi x_1/k$ for x_1, $\pi y_1/k$ for y_1, $\kappa\pi/U_0 k$ for κ, U/U_0 for U, V/U_0 for V, $\pi F_1/\rho U_0^2 k$ for F_1, and so on. We obtain

(i) (The vortex appreciably downstream).

$$\kappa = \frac{2U_0 k}{\pi}\frac{\sin\lambda \sin\frac{1}{4}\delta}{\sin(\lambda - \frac{1}{4}\delta)},$$

$$x_1 = \frac{k}{\pi}\left[\log\frac{1}{\epsilon} + (\tfrac{1}{2}\pi - \lambda + \tfrac{1}{4}\delta)\sin\delta + 2\sin^2\tfrac{1}{4}\delta\frac{\sin(\lambda+\frac{1}{4}\delta)}{\sin(\lambda-\frac{1}{4}\delta)} + 2\sin^2\tfrac{1}{2}\delta\log 2\right],$$

$$y_1 = \frac{k}{\pi}\left[-(\tfrac{1}{2}\pi - \lambda + \tfrac{1}{4}\delta)\cos\delta + \sin\delta\log\{2\sin(\lambda - \tfrac{1}{4}\delta)\} - 2\sin^2\tfrac{1}{4}\delta\frac{\cos(\lambda+\frac{1}{4}\delta)}{\sin(\lambda-\frac{1}{4}\delta)}\right],$$

$$Q_1 = U_0\frac{\sin(\lambda - \frac{1}{4}\delta)\cos\frac{1}{4}\delta}{\sin\lambda}, \qquad \alpha_1 = \tfrac{1}{2}\delta,$$

$$F_1 = -\rho 2\kappa\pi Q_1 = -\rho 2\pi\sin\tfrac{1}{2}\delta, \qquad \beta_1 = \tfrac{1}{2}\pi + \tfrac{1}{2}\delta. \tag{5.1}$$

When the vortex is also weak,

$$\delta = \frac{2\kappa\pi}{U_0 k}, \qquad \frac{1}{R_1} = \frac{\kappa\pi^2}{U_0 k^2}\left(\tfrac{1}{3} + \tfrac{1}{4}\tan^2\frac{\pi y_1}{k}\right),$$

$$\Delta U = \frac{\kappa\pi}{2k}\tan\frac{\pi y_1}{k}, \qquad \Delta V = \frac{\kappa\pi}{k},$$

$$Q_1 = U_0, \qquad \alpha_1 = \frac{\kappa\pi}{U_0 k},$$

$$F_1 = -\rho 2\kappa\pi U_0, \qquad \beta_1 = \tfrac{1}{2}\pi + \frac{\kappa\pi}{U_0 k}. \tag{5.2}$$

The formulae, apart from that for x_1, apply also to the case of a vortex in a stream with only free boundaries.*

* Cf. (4) § 6.

(ii) (The vortex near an edge of the orifice).

$$\delta = \left[\frac{2\kappa\pi}{U_0 k}\Phi_4(\alpha,\beta)\right]^{\frac{1}{2}} \tag{5.3}$$

The lower and upper free stream-lines are respectively

$$\psi = \delta\left(\tanh\frac{\pi s}{2k}\right)^{\frac{1}{2}} \quad \text{and} \quad \psi = \delta\left(\coth\frac{\pi s}{2k}\right)^{\frac{1}{2}}.$$

Also $$\pm x_1 = \frac{\kappa}{U_0}\Phi_1(\alpha,\beta), \qquad \pm\tfrac{1}{2}k - y_1 = \frac{\kappa}{U_0}\Phi_2(\alpha,\beta),$$

according as the vortex is near the lower or upper edge. Further

$$U_1 = U_0\Phi_3(\alpha,\beta), \qquad V_1 = \pm U_0\Phi_4(\alpha,\beta),$$
$$F_1\cos\beta_1 = \rho 2\kappa\pi U_0\Phi_4(\alpha,\beta), \qquad F_1\sin\beta_1 = \mp\rho 2\kappa\pi U_0\Phi_3(\alpha,\beta)$$

(the signs being determined as above).

The functions $\Phi_r\,(r = 1, 2, 3, 4)$ are those defined by equations (3.4).

(iii) (The vortex weak).

$$\sigma = \left|\cosh\frac{\pi z_1}{k}\right|,$$

$$\delta = \frac{\kappa\pi}{\sigma U_0 k}\left[\left(\sigma+\sinh\frac{\pi x_1}{k}\right)\left(\sigma+\cosh\frac{\pi x_1}{k}\right)\right]^{\frac{1}{2}},$$

$$\psi = \frac{\frac{1}{2}\delta(1-e^{-2\pi x/k})^{\frac{1}{2}}}{\sigma+\cosh\frac{\pi x_1}{k}}\,\frac{e^{\pi(x-x_1)/k}\left(\sigma+\cosh\frac{\pi x_1}{k}\right)\pm e^{\pi x_1/k}\sin\frac{\pi y_1}{k}}{\cosh\frac{\pi}{k}(x-x_1)\pm\sin\frac{\pi y_1}{k}},$$

$$\Delta U = \frac{\kappa\pi}{2\sigma k}\left[\sinh\frac{\pi x_1}{k}\tan\frac{\pi y_1}{k}+\frac{1}{\sigma}\sin\frac{\pi y_1}{k}\cos\frac{\pi y_1}{k}-\right.$$
$$\left.-2\sin\frac{\pi y_1}{k}\left(\frac{\sigma+\sinh\frac{\pi x_1}{k}}{\sigma+\cosh\frac{\pi x_1}{k}}\right)^{\frac{1}{2}}\right],$$

$$\Delta V = \frac{\kappa\pi}{2\sigma^2 k}\left[\sigma+\sinh\frac{\pi x_1}{k}\right]\cosh\frac{\pi x_1}{k},$$

$$\frac{1}{R_1} = \frac{\kappa\pi^2}{U_0 k^2}\left[\frac{1}{3}-\frac{1}{8}\left(1-\frac{2\sinh\frac{\pi x_1}{k}\sec^2\frac{\pi y_1}{k}}{\sigma}+\right.\right.$$
$$\left.\left.+\frac{\sinh^2\frac{\pi x_1}{k}\cosh^2\frac{\pi x_1}{k}-\sin^2\frac{\pi y_1}{k}\cos^2\frac{\pi y_1}{k}}{\sigma^4}\right)\right],$$

$$F_1 = -\rho 2\kappa\pi U_0, \qquad \beta_1 = \tfrac{1}{2}\pi+\frac{\Delta V}{U_0}. \tag{5.4}$$

When the vortex is symmetrically placed, the formulae are

$$\delta = \frac{\kappa\pi}{U_0 k}\left(2+2\tanh\frac{\pi x_1}{k}\right)^{\frac{1}{2}}, \qquad \psi = \frac{\delta(1-e^{-2\pi x/k})^{\frac{1}{2}}}{1+e^{2\pi(x_1-x)/k}},$$

$$\Delta U = 0, \qquad \Delta V = \frac{\kappa\pi}{2k}\left(1+\tanh\frac{\pi x_1}{k}\right),$$

$$\frac{1}{R_1} = \frac{\kappa\pi^2}{U_0 k^2}\left[\frac{1}{3}-\frac{1}{8}\left(1-\tanh\frac{\pi x_1}{k}\right)^2\right],$$

$$F_1 = -\rho 2\kappa\pi U_0, \qquad \beta_1 = \tfrac{1}{2}\pi + \frac{\Delta V}{U_0}. \tag{5.5}$$

All these equations, except that for ΔU, are correct to a second approximation.

6. Application to aerofoil theory

There are two applications: (*a*) to an aerofoil in the stream of a wind tunnel, (*b*) to an aerofoil in a free stream of limited breadth extending to infinity both upstream and downstream.

Putting Γ' for the circulation in the clockwise sense, we have

$$\Gamma' = -2\kappa\pi. \tag{6.1}$$

In all practical cases $\kappa\pi/kU_0$ is a small quantity, so that the problem falls within the purview of case (iii) (5.4), or of the case with free boundary and weak vortex.*

The stream is deflected downwards, say at an angle γ ($= -\delta$). Let L' denote the lift when the inclination of the direction of zero lift to the walls in (*a*), or the initial direction of the stream in (*b*), is α, and let C_L denote the lift coefficient of the aerofoil corresponding to an angle of incidence α. Applying the Kutta-Joukowski theorem, we have $L' = \rho\Gamma' U_1$, or, to a first approximation,

$$L' = \rho\Gamma' U_0, \tag{6.2}$$

This relation is not true exactly, as is the case in unlimited flow.

Now the effects of the limitation of motion on the flow around the aerofoil are

(i) the angle of incidence is increased by $\Delta_1\alpha$, say ($= \Delta V/U_0$): $\Delta_1\alpha$ is of the same order of magnitude as α, and so the correction due to this source may be considerable;

* (4) § 6.

(ii) the velocity of incidence is increased by amount ΔU: in the symmetrical case $\Delta U = 0$; even in the asymmetrical case $\Delta U/U_0$ is of order κ, so that the effect here is negligible compared with that in (i);

(iii) the stream incident on the aerofoil becomes curved. The curvature gives rise, according to Carafoli (**1**), to an increase in circulation of amount equivalent to a virtual increase in α given by

$$\Delta_2 \alpha = \frac{c}{4R_1}, \tag{6.3}$$

where c is the chord of the wing.

In practice, $\Delta_1 \alpha$ will be considerably greater than $\Delta_2 \alpha$, but the second may not be negligible. We shall write

$$\Delta\alpha = -\Delta_1 \alpha - \Delta_2 \alpha, \tag{6.4}$$

where $\Delta\alpha$ will now be a positive decrement. It is known that, to a sufficient order of accuracy,

$$C_L/\alpha = 2\aleph, \tag{6.5}$$

a constant for the aerofoil. Hence the lift coefficient to be taken into account here is $2\aleph(\alpha - \Delta\alpha)$, and

$$L' = \tfrac{1}{2}\rho c 2\aleph(\alpha - \Delta\alpha)U_0^2 = \tfrac{1}{2}\rho c C_L\left(1 - \frac{\Delta\alpha}{\alpha}\right)U_0^2. \tag{6.6}$$

Defining an apparent lift coefficient C_L' so that

$$L' = \tfrac{1}{2}\rho c C_L' U_0^2, \tag{6.7}$$

and

$$A = \frac{4k}{c}\frac{\Delta\alpha}{C_L'}, \tag{6.8}$$

we have

$$C_L' = C_L\left(1 - \frac{C_L'}{2\alpha}\frac{c}{2k}A\right), \tag{6.9}$$

or

$$\frac{C_L}{C_L'} = 1 + \aleph\frac{c}{2k}A,$$

giving the true lift coefficient in terms of the observed quantities. We also find, from (6.2), that, to a first approximation,

$$\Gamma' = \tfrac{1}{2}C_L' c U_0. \tag{6.10}$$

Also, if ΔC_D denote the correction to the observed drag coefficient taken as a positive decrement,

$$\Delta C_D = C_L' \cot\beta_1 = -C_L'\Delta_1\alpha. \tag{6.11}$$

We proceed to apply these formulae:

(*a*) *Aerofoil in wind tunnel.*

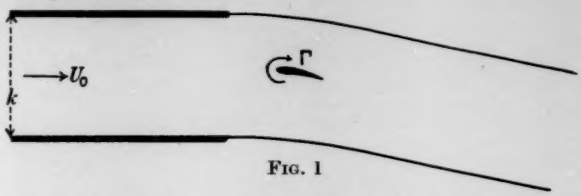

FIG. 1

(Symmetrical case.) The formulae of case (iii) give

$$\gamma = \frac{C'_L c}{2k}\left(\frac{1}{2}+\frac{1}{2}\tanh\frac{\pi x_1}{k}\right)^{\frac{1}{2}},$$

$$\Delta_1\alpha = -\frac{C'_L c}{8k}\left(1+\tanh\frac{\pi x_1}{k}\right),$$

$$\Delta_2\alpha = -\frac{\pi C'_L c^2}{16k^2}\left[\frac{1}{3}-\frac{1}{8}\left(1-\tanh\frac{\pi x_1}{k}\right)^2\right],$$

$$A = \frac{1}{2}\left(1+\tanh\frac{\pi x_1}{k}\right)+\frac{\pi c}{12k}\left[1-\frac{3}{8}\left(1-\tanh\frac{\pi x_1}{k}\right)^2\right]. \qquad (6.12)$$

With the usual arrangement it will generally suffice to take

$$A = \frac{1}{2}\left(1+\tanh\frac{\pi x_1}{k}\right)+\frac{\pi c}{12k}. \qquad (6.13)$$

(Asymmetrical case.) The formulae of case (iii) give

$$\gamma = \frac{C'_L c}{4\sigma k}\left[\left(\sigma+\sinh\frac{\pi x_1}{k}\right)\left(\sigma+\cosh\frac{\pi x_1}{k}\right)\right]^{\frac{1}{2}}, \qquad (6.14)$$

$$\Delta_1\alpha = -\frac{C'_L c}{8\sigma^2 k}\left[\sigma+\sinh\frac{\pi x_1}{k}\right]\cosh\frac{\pi x_1}{k},$$

$$\Delta_2\alpha = -\frac{\pi C'_L c^2}{16k^2}\left[\frac{1}{3}-\frac{1}{8}\left(1-\frac{2\sinh\frac{\pi x_1}{k}\sec^2\frac{\pi y_1}{k}}{\sigma}+\right.\right.$$

$$\left.\left.+\frac{\sinh^2\frac{\pi x_1}{k}\cosh^2\frac{\pi x_1}{k}-\sin^2\frac{\pi y_1}{k}\cos^2\frac{\pi y_1}{k}}{\sigma^4}\right)\right],$$

$$A = \frac{1}{2\sigma^2}\left[\sigma+\sinh\frac{\pi x_1}{k}\right]\cosh\frac{\pi x_1}{k}+\frac{\pi c}{12k}\left[1-\frac{3}{8}\left(1-\frac{2\sinh\frac{\pi x_1}{k}\sec^2\frac{\pi y_1}{k}}{\sigma}+\right.\right.$$

$$\left.\left.+\frac{\sinh^2\frac{\pi x_1}{k}\cosh^2\frac{\pi x_1}{k}-\sin^2\frac{\pi y_1}{k}\cos^2\frac{\pi y_1}{k}}{\sigma^4}\right)\right].$$

With the usual arrangement it will generally suffice to take

$$A = \frac{1}{2}\left(1+\tanh\frac{\pi x_1}{k}\right)+\frac{3}{2}\sin^2\frac{\pi y_1}{k}\left(1-\tanh\frac{\pi x_1}{k}\right)+\frac{\pi c}{12k}\left(1+\frac{3}{4}\tan^2\frac{\pi y_1}{k}\right). \tag{6.15}$$

(*b*) *Aerofoil in a free stream.*

FIG. 2

$$\gamma = \frac{C_L' c}{2k}, \qquad \Delta_1\alpha = -\frac{C_L' c}{4k},$$

$$\Delta_2\alpha = -\frac{\pi C_L' c^2}{16k^2}\left(\frac{1}{3}+\frac{1}{4}\tan^2\frac{\pi y_1}{k}\right), \qquad A = 1+\frac{\pi c}{12k}\left(1+\frac{3}{4}\tan^2\frac{\pi y_1}{k}\right). \tag{6.16}$$

7. Conclusion

The results deduced by the above method for the case of an aerofoil placed symmetrically with respect to the wind tunnel are

$$\frac{C_L}{C_L'} = 1+\aleph\frac{c}{4k}\left[1+\tanh\frac{\pi x_1}{k}+\frac{\pi c}{6k}\right],$$

and

$$\Delta C_D = \frac{C_L'^2 c}{8k}\left(1+\tanh\frac{\pi x_1}{k}\right). \tag{7.1}$$

These results are accurate to the second order. The corresponding results from Prandtl's method, given by Toussaint,* are seen to be of entirely different form. This appears to be due chiefly to the substitution, at one point in that analysis, of $\aleph$ for $C_L'/2\alpha$. We suggest that $\aleph = C_L/2\alpha$ leads to a better result.

It will be convenient to deal here also with another point, referred to in § 1. We note that no allowance is made in Toussaint's work for the effect of the curvature of the stream at the aerofoil, although the correction for this is found to have weight in the case of a wind

* (2) 4.21.

tunnel of closed working section. The present analysis shows that $\Delta_2\alpha/\Delta_1\alpha$ is of the order $c/4k$, so that, although the effect of curvature is small compared with that due to the alteration in the incident stream, it is not always negligible.

To compare the numerical values of the corrections given by the two methods, we have the following table, calculated for the typical case where $c/k = 0{\cdot}20$, and for various values of x_1/k.

	$\frac{x_1}{k} =$	0·10	0·20	0·30	0·40	0·50	0·60	∞
Present method	$\frac{C_L}{C'_L} =$	1·243	1·287	1·318	1·337	1·349	1·355	1·363
	$\frac{\Delta C_D}{C'^2_L} =$	0·033	0·039	0·043	0·046	0·048	0·049	0·050
Prandtl and Toussaint	$\frac{C_L}{C'_L} =$	1·117	1·238	1·341	1·415	1·442	1·491	1·527
	$\frac{\Delta C_D}{C'^2_L} =$	0·017	0·034	0·049*	0·060	0·066	0·071	0·076

It is seen that the latter corrections are in deficit for small values of x_1/k and in excess for larger values. The discrepancy ultimately increases with x_1: it also increases with c/k. The point at which the formulae for C_L agree is approximately the same as that at which those for ΔC_D agree. The formulae proposed above do not lead to zero corrections when $x_1 = 0$ as do those of (**2**).

We note further, in our treatment of problem (*a*), that, provided that the unabridged formula for A be used, there is no restriction on the value of x_1. It is thus possible to put $x_1 = -\infty$, when the stream is entirely enclosed between two fixed walls. Corrections are thus obtained for the wind tunnel of closed working section, which in the symmetrical case are in agreement with the results of the Prandtl method, amended by the substitution of $\aleph$ for $C_L/2\alpha$ instead of $C'_L/2\alpha$. We find the increase of lift quoted by Toussaint [(**2**), 301] and not the decrease obtained by Sasaki (**3**), who, however, replaced the aerofoil by a flat plate.

It is again a pleasure, in closing, to acknowledge the supervision of this work by Dr. R. C. Knight, and the advantage of his suggestions.

* The numerical results for the drag, given by Toussaint [(**2**), 305], do not appear to follow from the formulae given.

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ON THE LOBATCHEWSKIAN TRIGONOMETRY OF A STATIC SUBSTRATUM

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[Received 7 September 1939]

1. Introduction

It is generally agreed that the most novel and significant contribution which Milne has made to cosmological theory is his discovery of the two families of time-scales, t and τ.

The family of t-scales comprises all clocks, kept by any particular fundamental observer, in terms of which the substratum is described as uniformly expanding from a point-origin at the singular epoch $t = 0$. The family of τ-scales, on the other hand, comprises all clocks kept by the same observer, in terms of which the substratum is described as static. Given any one t-clock Milne has shown that we can associate with it a unique τ-clock which both reads the same epoch and beats at the same rate as the t-clock at an instant t_0, which we can identify with the present. The relation between these two clocks is given by*

$$\tau = t_0 \log(t/t_0) + t_0. \tag{1}$$

It was further shown by Milne that, if the observer using a particular t-clock plots his observations in a private Euclidean space and then transforms by (1) to a τ-clock, he generates a public hyperbolic space of constant negative curvature $-(ct_0)^{-2}$. This result was proved by constructing the metric of this space.†

It is the object of the present note to obtain this result in a different manner. I do this because I wish to point out an interesting parallelism between Milne's discovery of the existence of a public position-space, for relatively stationary observers, and the apparently neglected theorem, due to Sommerfeld and, independently, to Varićak, establishing the existence of a public kinematic-space for observers in uniform relative motion.‡ Either result can be derived from the other by means of a simple formula, due to Milne, correlating a certain function of the speed assigned to a particle by an

* Milne, *Proc. Roy. Soc.* A, 158 (1937), 324.

† Ibid. 159 (1937), 171.

‡ Sommerfeld, *Verh. deutschen Phys. Ges.* 11 (1909), 577; Varićak, *Phys. Zeits.* 11 (1910), 93, 287, 586.

observer using a given t-clock and the distance assigned to the same particle by the same observer at the corresponding instant according to the τ-clock given by equation (1). For completeness I first establish this formula *de novo* and correlate Milne's transformations with certain transformations due to Lewis. This work is essentially one-dimensional because we need only consider an event in alinement with two observers. I then pass to the general case and give a new proof of Sommerfeld and Varićak's theorem, which is more concise than either of theirs because it is based on the use of 4-vectors. In this way I show how the distances and orientations of any event, according to two observers using congruent τ-clocks, can be correlated and I also establish the invariance of the τ-epoch assigned to any event by these observers, without making the usual appeal to the geometry of 'cross-country' light paths.

2. Two equivalent observers and a collinear event

It is well known that the Lorentz transformation correlating the distances and epochs assigned by two observers,* using congruent t-clocks, to a collinear event can be regarded geometrically as the rotation through an imaginary angle of the axes of space-time co-ordinates. This is because the transformation can be written in the form

$$x' = x\cosh\omega - ct\sinh\omega, \qquad ct' = ct\cosh\omega - x\sinh\omega, \tag{2}$$

where $$\cosh\omega = \frac{1}{(1-V^2/c^2)^{\frac{1}{2}}}, \qquad \sinh\omega = \frac{V/c}{(1-V^2/c^2)^{\frac{1}{2}}}, \tag{3}$$

and hence $$\tanh\omega = V/c. \tag{4}$$

The term $\omega = \tanh^{-1}V/c$ has been called the relative *rapidity* of the observers and was introduced by Robb† because, in the Special Theory of Relativity, the rapidities, and not the velocities, of collinear uniformly moving observers are additive.

Recently, Lewis‡ has drawn attention to an alternative way of regarding the Lorentz transformation as a rotation of axes. Lewis

* In uniform relative motion of speed V, from coincidence at zero epoch.

† Robb, *Optical Geometry of Motion* (1911), 9; *Geometry of Time and Space* (1936), 406; Eddington, *Mathematical Theory of Relativity* (1930), 22.

‡ Lewis, *Phil. Mag.* 20 (1935), 1096. His notation has been slightly altered so as to lead to no confusion between his symbols and mine.

introduces two auxiliary variables σ, ϕ, analogous to polar coordinates, and writes the transformation in the form

$$\left.\begin{aligned} x &= c\sigma\sinh\phi, & t &= \sigma\cosh\phi \\ x' &= c\sigma\sinh(\phi-\omega), & t' &= \sigma\cosh(\phi-\omega) \end{aligned}\right\}, \tag{5}$$

where ω is the relative rapidity. We see immediately that

$$\sigma = (t^2-x^2/c^2)^{\frac{1}{2}}, \tag{6}$$

and

$$\phi = \tanh^{-1}\frac{x}{ct} = \frac{1}{2}\log\frac{t+x/c}{t-x/c}.$$

The parameter σ is invariant, but the parameter ϕ transforms into ϕ', where

$$\phi' = \phi-\omega. \tag{7}$$

Milne* has shown that, if (x, t) are the distance and epoch assigned to an event by an observer using a t-clock and if (λ, τ) are the distance and epoch assigned to the same event by the same observer using the τ-clock given by (1), then the following relations exist:

$$\left.\begin{aligned} t &= t_0\, e^{(\tau-t_0)/t_0}\cosh\frac{\lambda}{ct_0} \\ x &= ct_0\, e^{(\tau-t_0)/t_0}\sinh\frac{\lambda}{ct_0} \end{aligned}\right\}. \tag{8}$$

Consequently, Lewis's parameters σ, ϕ become

$$\left.\begin{aligned} \sigma &= (t^2-x^2/c^2)^{\frac{1}{2}} = t_0\, e^{(\tau-t_0)/t_0} \\ \phi &= \tanh^{-1}\frac{x}{ct} = \frac{\lambda}{ct_0} \end{aligned}\right\}. \tag{9}$$

Furthermore, since on the t-scale the observers' relative speed of recession V is given by

$$V = \frac{x}{t} = c\tanh\frac{\lambda}{ct_0},$$

it follows that, if, on the τ-scale, the same observers' constant distance apart is Λ, then, as Milne has shown,

$$V = c\tanh\frac{\Lambda}{ct_0}. \tag{10}$$

Consequently, if ω is the relative rapidity of the observers using the t-scale, then

$$\omega = \frac{\Lambda}{ct_0}. \tag{11}$$

* Milne, *Proc. Roy. Soc.* A, 159 (1937), 177.

We thus deduce from (7), (9), and (11) that, if the distances assigned to an event on their common radius vector by two observers, distant Λ apart, using congruent τ-scales are λ and λ' respectively, then

$$\lambda' = \lambda - \Lambda. \tag{12}$$

Consequently, the τ-scale distances between the observers and a collinear event are additive. Similarly, from (9) it follows that the invariance of σ implies that the respective conventional epochs τ and τ', assigned by the observers to a collinear event, are identical.

3. Two equivalent observers and any event

We now pay attention to orientation and consider two observers and a non-collinear event.

First, we observe that the invariance of the conventional epoch assigned by two observers, using congruent τ-scales, to a collinear event can be immediately extended to the case of a non-collinear event, without making any appeal to the geometry of 'cross-country' light paths.* This is possible because we can always locate a fundamental observer at any event in the substratum. The conventional τ-epoch assigned to the event by any other fundamental observer is identical with this particular observer's own τ-scale reading at the epoch concerned. Consequently, all fundamental observers assign the same τ-epoch to any given event.†

We now consider the relations between the radial distances and orientations of any event relative to two observers using congruent τ-clocks. Milne has found that there exists a public hyperbolic space of constant negative curvature $-1/(ct_0)^2$, it being assumed in the first place that each observer using his original t-scale correlates his observations with those of the other observer by means of the three-dimensional Lorentz formulae. We now see, in view of relation (11), that Milne's result is formally identical with the theorem of Sommer-

* For the necessity of such an appeal, or its equivalent, to establish the Lorentz formulae in space of two or more dimensions *vide* Milne, *Relativity, Gravitation, and World-Structure* (1935), § 36.

† Incidentally, if we transform back to the t-scale, given by inverting (1), we find that the well-known invariance of the quadratic form, $X = t^2 - \mathbf{P}^2/c^2$, where t is the conventional epoch assigned to an event at $\mathbf{P}$, according to any member of a pencil of equivalent observers uniformly expanding from a point origin at zero epoch, is independent of the validity of the three-dimensional Lorentz formulae and is purely a consequence of the one-dimensional formulae. This point is not usually made clear in writings on the subject.

feld and Varićak, that in the Minkowski diagram the trigonometry of rapidities is Lobatchewskian. I proceed to establish this theorem *de novo*, by means of 4-vector analysis. I find it convenient to consider three observers, instead of simply two observers and an event. As we have already remarked, we can always locate a fundamental observer at any event in the substratum.

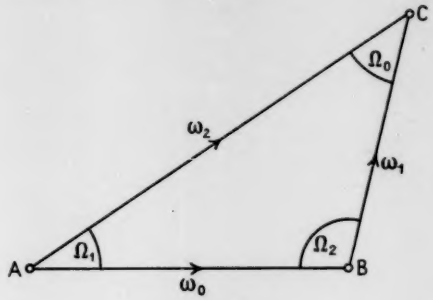

FIG. 1

Let A, B, C be any three non-collinear members of a pencil of equivalent observers in uniform relative motion from a point-origin at zero epoch, each observer plotting his observations in a Euclidean private space and correlating his observations with those of the other observers by means of the three-dimensional Lorentz formulae. Let the constant angle subtended at A, according to A, by B and C, be Ω_1; the constant angle subtended at B, according to B, by A and C, be Ω_2; and the constant angle subtended at C, according to C, by A and B, be Ω_0. Let the relative rapidities of A and B, B and C, C and A be ω_0, ω_1, ω_2 respectively. Then the triangle, constructed in Fig. 1, with sides ω_0, ω_1, ω_2 and angles Ω_0, Ω_1, Ω_2 will obey Lobatchewskian trigonometry if

$$\cosh\omega_2 = \cosh\omega_0\cosh\omega_1 - \sinh\omega_0\sinh\omega_1\cos\Omega_2, \tag{13}$$

and if

$$\frac{\sin\Omega_0}{\sinh\omega_0} = \frac{\sin\Omega_1}{\sinh\omega_1} = \frac{\sin\Omega_2}{\sinh\omega_2}. \tag{14}$$

If $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$ be the relative velocities of A and B, B and C, C and A respectively, then, by formulae (3),

$$\cosh\omega_0 = \frac{1}{(1-\mathbf{V}_0^2/c^2)^{\frac{1}{2}}}, \qquad \sinh\omega_0 = \frac{|\mathbf{V}_0|/c}{(1-\mathbf{V}_0^2/c^2)^{\frac{1}{2}}} \tag{15}$$

and similarly for $\cosh\omega_1$, $\sinh\omega_1$ and $\cosh\omega_2$, $\sinh\omega_2$. Consequently,

$$\cosh\omega_0\cosh\omega_1-\sinh\omega_0\sinh\omega_1\cos\Omega_2=\frac{1-\mathbf{V}_0.\mathbf{V}_1/c^2}{(1-\mathbf{V}_0^2/c^2)^{\frac{1}{2}}(1-\mathbf{V}_1^2/c^2)^{\frac{1}{2}}}. \tag{16}$$

The expression on the right-hand side of this identity is simply *the scalar product of the two four-vectors*,

$$\left(\frac{\mathbf{V}_0/c}{Y_0^{\frac{1}{2}}},\frac{1}{Y_0^{\frac{1}{2}}}\right) \quad \text{and} \quad \left(\frac{\mathbf{V}_1/c}{Y_1^{\frac{1}{2}}},\frac{1}{Y_1^{\frac{1}{2}}}\right),$$

where $\qquad Y_0^{\frac{1}{2}}=(1-\mathbf{V}_0^2/c^2)^{\frac{1}{2}}, \qquad Y_1^{\frac{1}{2}}=(1-\mathbf{V}_1^2/c^2)^{\frac{1}{2}}.$

It is therefore invariant under all Lorentz transformations. If we regard A and C as fixed observers, it follows that this scalar product has the same value howsoever we choose B from the pencil of equivalent observers considered. In particular, if we identify B with C, this scalar product reduces to $(1-\mathbf{V}_2^2/c^2)^{-\frac{1}{2}}$. It follows that

$$\frac{1}{(1-\mathbf{V}_2^2/c^2)^{\frac{1}{2}}}=\frac{1-\mathbf{V}_0.\mathbf{V}_1/c^2}{(1-\mathbf{V}_0^2/c^2)^{\frac{1}{2}}(1-\mathbf{V}_1^2/c^2)^{\frac{1}{2}}}, \tag{17}$$

and hence that

$$\cosh\omega_2=\cosh\omega_0\cosh\omega_1-\sinh\omega_0\sinh\omega_1\cos\Omega_2.$$

Formula (13) is, therefore, established.

To verify (14) we observe that, since

$$\sinh\omega_2=\frac{|\mathbf{V}_2|/c}{(1-\mathbf{V}_2^2/c^2)^{\frac{1}{2}}},$$

it follows that $\qquad \sinh\omega_2\sin\Omega_1=\dfrac{v_2/c}{(1-\mathbf{V}_2^2/c^2)^{\frac{1}{2}}},$

where v_2 is the component of $\mathbf{V}_2$, orthogonal to the direction AB. Since

$$\left(\frac{\mathbf{V}_1/c}{Y_1^{\frac{1}{2}}},\frac{1}{Y_1^{\frac{1}{2}}}\right) \quad \text{and} \quad \left(\frac{\mathbf{V}_2/c}{Y_2^{\frac{1}{2}}},\frac{1}{Y_2^{\frac{1}{2}}}\right)$$

are 4-vectors, we see that, if y, y' are the coordinates assigned to C by A, B respectively in the direction orthogonal to AB, then, since $y'=y$,

$$\frac{v_1}{(1-\mathbf{V}_1^2/c^2)^{\frac{1}{2}}}=\frac{v_2}{(1-\mathbf{V}_2^2/c^2)^{\frac{1}{2}}}, \tag{18}$$

where v_1 is the component of $\mathbf{V}_1$ orthogonal to AB. Consequently,

$$\sinh\omega_2\sin\Omega_1=\sinh\omega_1\sin\Omega_2,$$

and hence $$\frac{\sin\Omega_1}{\sinh\omega_1} = \frac{\sin\Omega_2}{\sinh\omega_2}.$$

In this way formulae (14) can be established.

We have thus proved that a substratum of equivalent observers uniformly expanding from a point source possesses a public kinematic space, since the trigonometry of rapidity vectors is that of a hypersphere of radius i. We can draw the following conclusions:

(I) Any two rapidity vectors* can be compounded by the triangle rule of hyperbolic space. Incidentally we note that these rapidity vectors are not free vectors but line vectors. There is consequently no parallelogram rule.

(II) The resultant of any number of rapidity vectors arranged in a chain in hyperbolic space is given by the geodesic drawn from the beginning to the end of the chain.

(III) Finally, by equation (11), we see that, when the observers regraduate their congruent clocks from t-time to τ-time by formula (1), there exists a public position space, which is hyperbolic and of constant negative curvature $-(ct_0)^{-2}$. The relations between the radial distances and orientations assigned to any event by any two observers of the system are thus given by the appropriate formulae of Lobatchewskian trigonometry.

* As we are primarily interested in equivalent observers, we have considered only rapidity vectors corresponding to motions from a common point origin. The proof, however, can be applied step by step to rapidity vectors in general as in the Special Theory of Relativity.

ON A CONJECTURE OF GILLIS

By T. VIJAYARAGHAVAN (*Dacca*)

[Received 8 November 1939]

SUPPOSE that $f(x)$ is continuous and periodic and that the line $y = mx+c$, where m and c are finite constants and $m \neq 0$, meets the curve $y = f(x)$ at a finite number of points only. Let C denote the curve $y = f(x)$, and L the line $y = mx+c$. I prove below that there exist finite constants m' and c' ($m' \neq 0$) such that $y = m'x+c'$ meets C at one point only. This result confirms a conjecture* of Gillis that, if C be such that every non-vertical straight line meets C in at least two points, then every such straight line meets C at an infinity of points. The existence of continuous periodic functions that give rise to such curves has been established by Gillis.†

Firstly, let $m > 0$. It is plain that L is on both sides of C, and ultimately is above C. Since by our hypothesis we have that L meets C at a finite number of points only, it will follow that there is a number ξ, and a positive number η such that

$$f(\xi) = m\xi+c, \tag{1}$$

$$f(\xi-\delta) > m(\xi-\delta)+c = f(\xi)-m\delta, \tag{2}$$

$$f(\xi+\delta) < m(\xi+\delta)+c = f(\xi)+m\delta \tag{3}$$

whenever $0 < \delta \leqslant \eta$.

Now, let H be the difference between the upper and lower bounds of $f(x)$. (4)

We consider $y = m'x+c'$ where m' and c' are determined so as to satisfy the following conditions:

$$m' > m > 0, \tag{5}$$

$$m'\eta > H, \tag{6}$$

$$m'\xi+c' = f(\xi). \tag{7}$$

We easily verify that

$$m'(\xi-\delta)+c' < f(\xi-\delta) \tag{8}$$

and

$$m'(\xi+\delta)+c' > f(\xi+\delta) \tag{9}$$

whenever $\delta > 0$. From (7), (8), (9) it follows that the straight line $y = m'x+c'$ meets C in one point only. We argue similarly if $m < 0$.

* See above, p. 154.

† Loc. cit. 151–4.

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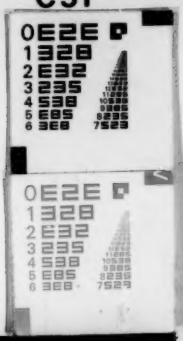
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